# NEW AND COMPLETE

# SYSTEM OF ALGEBRA:

## OR, SPECIOUS ARITHMETIC.

COMPREHENDING

ALL THE FUNDAMENTAL RULES AND OPERATIONS OF THAT SCIENCE,

*CLEARLY EXPLAINED AND DEMONSTRATED,*

WITH THE

RESOLUTION OF ALL KINDS OF EQUATIONS,

ILLUSTRATED AND EXEMPLIFIED

IN THE SOLUTION OF A VAST VARIETY OF THE MOST CURIOUS AND INTERESTING QUESTIONS.

*FOR THE USE OF SCHOOLS.*

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# ELEMENTS OF SPECIOUS ARITHMETICK.

## CHAP. I.

*Of the analytick Method of expreſſing Problems by Equations, and of the Reſolution of Equations of the firſt Degree.*

AMONGST the different Problems which employed the firſt Mathematicians called Analyſts, I chuſe the following, as the moſt proper to ſhew how they formed the Science ſtyled ſpecious Arithmetick.

I.

*To divide a Sum, for Example, £890 between three Perſons, in ſuch a Manner that the firſt may have £180 more than the ſecond, and the ſecond £115 more than the third.*

Example of a Problem ſimilar to thoſe which the firſt Analyſts might have propoſed to themſelves.

It is thus I imagine a Perſon would have argued, who, without the leaſt Tincture of ſpecious Arithmetick, attempted to ſolve this Problem.

It is manifeſt that if one of the three Parts was known, the other two would be immediately diſcovered; let us ſuppoſe, for Example, the third which is the leaſt to be known, we muſt add £115 to it, and this Sum will be the Value of the ſecond; to obtain afterwards the firſt Part, we muſt add £180 to this ſecond, which comes to the ſame as if we added £180 more £115 or £295 to the third.

Solution of this problem ſuch as might be found without ſpecious Arithmetic.

Let therefore this third Part be what it will, we know that this Part, more itſelf together with £115, more itſelf again together with £295 ſhould make a Sum equal to £890.

From whence it follows that the Triple of the leaſt Part, more £115, more £295, or more £410, is equal to £890.

But, if the Triple of the Part ſought more £410 be equal to £890, this Triple of the Part ſought muſt be leſs than £890 by £410, therefore it is equal to £480, therefore the leaſt Part is equal to £160. The ſecond will conſequently be £275, and the firſt or greateſt £450.

It is probable the firſt Analyſts argued in this Manner when they propoſed to themſelves Queſtions of this Nature, without doubt in proportion as they advanced in the Solution of a Problem, they burthened their Memories with all the Arguments which had conducted them to

the Point they had arrived at, and when the Problems were not more complicated than the foregoing, it was no difficult Matter; but as ſoon as their Reſearches preſented a greater Number of Ideas to be retained, they were under the Neceſſity of having recourſe to a more conciſe Method of expreſſing themſelves, and of employing ſome ſimple Symbols, by Means of which however advanced they were in the Solution of a Problem, they might perceive at one View what they had done and what remained for them to do. Now the Kind of Language they imagined for this Purpoſe, is called ſpecious Arithmetick.

II.

Algebraic Method of expreſſing the foregoing Problem

To explain the Principles of this Science more clearly, we will reſume the ſame Queſtion, write down in Words the Arguments which the Analyſt employs to ſolve his Problem, and in analytick Symbols what is requiſite to aſſiſt his Memory.

The leaſt or third Part, be it what it will, I denote by one Letter, for Example by $x$.

The Sign + denotes addition.

The ſecond conſequently will be $x$ more 115, which I denote thus, $x + 115$ employing the Sign + which ſignifies *more* to expreſs the Addition of the two Quantities between which it is placed.

As to the firſt Part or greateſt, ſince it exceeds the ſecond by 180, it will be expreſſed by $x + 115 + 180$.

Adding thoſe three Parts we will have $3x + 115 + 115 + 180$, or when reduced $3x + 410$.

The Sign = deſigns Equality.

But this Sum of the three Parts ſhould be equal to 890, which I expreſs thus $3x + 410 = 890$, employing the Symbol = which ſignifies *equal* to denote the Equality of the two Quantities between which it is placed.

An equation is the equality of two quantities. To ſolve an equation is to find the value of the unknown quantity it includes.

The Queſtion therefore by this Computation is changed into another, where it is required to find a Quantity, the Triple of which being added to 410 makes 890. To find the Reſolution of ſimilar Queſtions, is what is underſtood by ſolving an Equation, the Equation in the preſent Caſe is $3x + 410 = 890$ it is ſo called, becauſe it indicates the Equality of two Quantities, to ſolve this Equation, is to find the Value of the unknown Quantity $x$ from this Condition that its Triple more 410 makes 890.

III.

Reſolution of the equation which expreſſes the foregoing Problem. The ſymbol — denotes ſubtraction.

To ſolve this Equation the Analyſt argues and writes down his Arguments as follows. The Equation to be ſolved $3x + 410 = 890$, teaches us that we are to add 410 to $3x$ to make up the Sum 890, wherefore $3x$ are leſs than 890 by 410, which I expreſs thus $3x = 890 - 410$ employing the Sign — which ſignifies *leſs* to denote that the Quantity which it precedes ſhould be ſubtracted from that which it follows.

From this new Equation $3x = 890 - 410$, we deduce, by ſubtracting in effect 410 from 890, this other Equation $3x = 480$.

But if three $x$ be equal to 480, one $x$ will be the third Part of 480 or 160, which I write down thus, $x = \frac{480}{3} = 160$, and the Queſtion is ſolved, ſince it ſuffices to know one of the Parts to diſcover the reſt.

IV.

If we were deſirous of ſolving the Queſtion by inveſtigating the greateſt Part firſt, we might effect it in like Manner. Another ſolution of the foregoing Problem.

Let this firſt Part be expreſſed by $y$, the ſecond having 180 leſs, will be $y - 180$, and the third having 115 leſs than the ſecond, will be $y - 180 - 115$.

Now the Sum of thoſe three Quantities is $3y - 180 - 180 - 115$, that is, $3y - 475$.

But this Sum ſhould be equal to 890.

We have therefore the Equation $3y - 475 = 890$, by which we learn that $3y$ exceeds 890 by 475, ſince we muſt ſubtract 475 from $3y$ to obtain 890. Wherefore $3y = 890 + 475$, or $3y = 1365$. Wherefore $y$ or the greateſt Part $= 455$, as found above.

V.

If in the Problem it was propoſed to divide a Sum greater or leſs than the one employed, and that the Differences were Numbers different from thoſe made uſe of, it is manifeſt it would be ſolved after the ſame Manner. Let us ſuppoſe, for Example, the Problem to have been expreſſed thus.

*To divide* 9600 *between four Perſons, in ſuch a Manner that the firſt may have* 300 *more than the ſecond, and the ſecond* 250 *more than the third, and the third* 200 *more than the fourth.* Another example of the foregoing Problem.

We would have argued after the following Manner: Let the fourth Part be $x$, the third will be $x + 200$, the ſecond $x + 200 + 250$, and the firſt $x + 200 + 250 + 300$.

But the Sum of all thoſe Parts ſhould be equal to 9600, we have therefore the Equation $4x + 1400 = 9600$.

To ſolve this Equation, I obſerve, as in the foregoing, that if to $4x$ 1400 is to be added to make it equal to 9600, it muſt be equal to what remains of 9600 when 1400 has been ſubducted from it, which is wrote down thus, $4x = 9600 - 1400$, or $4x = 8200$.

But if four $x$ be equal to 8200, one $x$ will be equal to the fourth Part of 8200, that is, $x = \frac{8200}{4} = 2050$, the leaſt Part $x$ being known, the other Parts will be immediately diſcovered, the third $= 2250$, the ſecond $= 2500$, and the firſt 2800.

VI.

The Problem may be ſtill varied, and yet ſtill depend upon the ſame Principles; Let us ſuppoſe, for Example, that it was expreſſed thus. Third example of the foregoing Problem.

*To divide* 5500 *into two Parts, ſo that the firſt may have a third more than the ſecond, together with* 180.

It would be ſolved in the following Manner: Let the ſecond Part be $x$, the firſt will be $x + \frac{1}{3}x + 180$. Now as their Sum ſhould be equal to 5500, we will have the Equation $2x + \frac{1}{3}x + 180 = 5500$.

To ſolve this Equation, I firſt add $2x$ & $\frac{1}{3}x$ together, and their Sum will be $\frac{7}{3}x$, becauſe two Integers is equal to ſix thirds, and conſequently thoſe two Integers with one third make up ſeven thirds. Wherefore the foregoing Equation is reduced to $\frac{7}{3}x + 180 = 5500$, which by the ſame Reaſoning employed in the foregoing Examples will become $\frac{7}{3}x = 5500 - 180$, or $\frac{7}{3}x = 5320$.

Now if the third Part of $7x$ be 5320, the whole $7x$ will be three Times that Sum, which I expreſs thus, $7x = 5320 \times 3$ employing the Sign $\times$ to denote the Multiplication of the two Quantities it ſeparates.

The ſign $\times$ denotes multiplication.

Afterwards, inſtead of $7x = 5320 \times 3$, it ſuffices to write $7x = 15960$, which reſults on multiplying in Effect 5320 by 3.

And by the Means of this new Equation we have $x = \frac{15960}{7} = 2280$, Value of the ſecond Part.

The firſt Part will be readily found, ſince we have only to add to this Quantity 2280 its third Part 760 and 180 more, agreeable to the Conditions, and we will have 3220 for this firſt Part.

Beginners may exerciſe themſelves in varying ſtill more the Expreſſions of the foregoing Problem, and ſolving it in the different Caſes they may imagine, the Readineſs they will acquire will recompence them ſufficiently for their Trouble. For their further Aſſiſtance, I ſhall propoſe another Problem, in ſeveral Reſpects ſimilar to the former.

VII.

A new problem of the ſame nature with the foregoing.

*Three Merchants form a Society, the firſt furniſhes* £17000, *the ſecond* £13000, *the third* £10000; *but as they have Need of a Perſon to tranſact the Buſineſs attending their Commerce, he who furniſhed* £10000, *is ſatisfied to take on himſelf this Trouble, on Condition of being allowed* 3 *per Cent on the whole Profit that will ariſe more than the reſt: It happens that this Profit amounts to* £100000, *it is required to determine the Share of each.*

Let the Part of the firſt be $x$.

The ſecond having furniſhed leſs in the Ratio of 13 to 17 ſhould have a Sum leſs in this ſame Ratio, that is, only $\frac{13}{17}x$.

The third, ſuppoſing he received only a Sum proportioned to what he furniſhed, would have the ten 17ths of the firſt, but as he is to have beſides 3 per Cent upon the whole Profit, that is £3000 his Share will be $\frac{10}{17}x + 3000$.

And as the Sum of thoſe three Parts ſhould be £100000, we will have $x + \frac{13}{17}x + \frac{10}{17}x + 3000 = 100000$, or $x + \frac{13}{17}x + \frac{10}{17}x = 97000$.

To diſengage the unknown Quantity in this Equation, I obſerve, that $x + \frac{13}{17}x + \frac{10}{17}x$, or $\frac{17}{17}x + \frac{13}{17}x + \frac{10}{17}x$ is the ſame as $\frac{40}{17}x$;

wherefore $\frac{40}{17}x = 97000$, or $40.x = 97000 \times 17$, or $40x = 1649000$, or $x = \frac{1649000}{40} = 41225$.

The Share of the first being found, that of the second expressed by $\frac{13}{17}x$, will be $\frac{13}{17} \times 41225$, that is 31525, that of the third expressed by $\frac{10}{17}x + 3000$ will be $\frac{10}{17} \times 41225 + 3000 = 27250$.

VIII.

By those two Problems the Readers may form some Notion of Specious Arithmetick, they learn that in general the Solution of a Problem consists of two Parts, in the first the Analyst expresses by a Letter as $x$ or $y$, &c. the unknown Quantity sought, or one of those which when known, serves to determine the rest, he afterwards endeavours to arrive at an Equation including the unknown Quantity, which is effected by expressing the same Quantity two different Ways.

The analytick solution of a problem consists of two parts.

In the second it is Question of bringing the unknown Quantity to one Side of the Equation, leaving only known Quantities on the other.

In the first the problem is expressed by an equation.

The first of those two Parts is not easily reduced to precepts intelligible to Beginners, and perhaps can be learned only by Examples.

In the second the equation is solved.

As to the second Part, it may be explained with much greater Ease, after a general Manner.

IX.

In the Questions which we have already solved, we arrived at Equations, in which the unknown Quantity was affected only by the Multiplication or Division of known Numbers; these Kinds of Equations are called Equations of the first Degree, such are $2x - 10 = 56$, $\frac{2}{3}x + 15 = x - \frac{1}{5}x + 30$, &c. and the Problems which lead to those Equations, are called Problems of the first Degree.

Equations of the first degree are those in which the unknown quantity is only multiplied or divided by known quantities.

They are so called to distinguish them from those in which the unknown Quantity is raised to the Square, Cube, &c. which are said to be of the second Degree if the unknown Quantity be squared, &c.

Let it be proposed, for Example, to find a Number, the Triple of which being added to its Square, will be 65, the Problem to be solved in this Case is of the second Degree, and the Equation $3x + xx = 65$, (in which $xx$ denotes the Square of $x$) which expresses the Conditions of this Problem is an Equation of the second Degree.

The first Analysts could not have attained to the Resolution of those Equations till they had been a long Time exercised in solving Equations of the first Degree. We shall therefore proceed to investigate the Rules which those require.

X.

To find them, let us first resume the Equation $4x + 1400 = 9600$ treated Art. v. which is composed of three Terms, $4x$, 1400, 9600 (thus are called all the Parts of an Equation separated from each other by the Signs + or —) and let us observe, that by the same Reasoning, by which we deduced $4x = 9600 - 1400$, we may in all Kinds of

The terms of an equation are its parts seperated by the

signs + or — Equations take any Term whatſoever preceded by the Sign + and carry it over to the other Side of the Sign = by giving it the Sign —. If we had for Example, $50 + \frac{10}{3}x = 5x + 30$, we may carry the Term $\frac{10}{3}x$ over to the other Side, giving it the Sign — and write down the Equation thus, $50 = 5x + 30 - \frac{10}{3}x$; for we may ſay, as in Art. v. that ſince $\frac{10}{3}x$ is to be added to 50 to make it equal to the Quantity $5x + 30$, therefore 50 muſt be leſs than $5x + 30$ by the Quantity $\frac{10}{3}x$, that is, it muſt be equal to $5x + 30 - \frac{10}{3}x$.

We may in like Manner as we found, Art. III. that the Equation $3y - 475 = 890$ was changed into $3y = 890 + 475$, we may, I ſay, perceive that in general, the Terms which have the Sign — on one Side of the Sign of Equality, may be carried over to the other, by giving them the Sign +. If we had, for Example, $32 - 6x = 9x + 119$, we would conclude that $32 = 6x + 9x + 119$. For if 32 ſhould be diminiſhed by $6x$ to equal $9x + 119$, it muſt exceed this Quantity by $6x$, that is, it is equal to $6x + 9x + 119$.

XII.

Any term may be carried over from one ſide of the equation to the other on changing its ſign. The members of an equation are its two parts ſeparated by the ſign =

Hence there reſults this general Principle for all Equations, that we may carry over what Terms we pleaſe from one Side of the Equation to the other, provided we change their Signs. Now this Principle is of infinite Uſe, in as much as it ſaves a great deal of Reaſoning.

XIII.

By its Means we may change any Equation into another, in which there will be on one Side of the Sign =, that is, in one of the Members of the Equation, the Terms affected by $x$, and on the other Side of the Sign =, that is, in the other Member of the Equation the Terms which are entirely known.

Let, for Example, the Equation $8x + 30 = \frac{5}{4}x + 250$ be given; I deduce from it $8x - \frac{5}{4}x = 250 - 30$; in like Manner from $60 - \frac{5}{4}x = 250 - \frac{7}{3}x$, I deduce $\frac{7}{3}x - \frac{5}{4}x = 250 - 60$, and ſo of any other Equation.

XIII.

When after the neceſſary Tranſpoſitions we have carried over all the Terms affected by $x$ to one Side of the Equation, and the known Terms to the other; what moſt naturally occurs, is to reduce each of the two Members of the Equation to its moſt ſimple Expreſſion. Let, for Example, the Equation $8x - \frac{5}{4}x = 250 - 30$ be given; I reduce it to $\frac{29}{4}x = 220$, by taking away in effect 30 from 250, and $\frac{5}{4}x$ from $8x$, or from $\frac{29}{4}x$ which is equal to it. The Equation $\frac{7}{3}x - \frac{5}{4}x = 250 - 60$ will be reduced to $\frac{13}{12}x = 190$, becauſe $\frac{7}{3}x$ and $\frac{5}{4}x$ brought to the ſame Denomination become $\frac{28}{12}x$ and $\frac{15}{12}x$ whoſe Difference is $\frac{13}{12}x$, and taking away 60 from 250 there remains 190.

XIV.

By ſimilar Reductions all Equations of the firſt Degree, however com-

poſed, will be changed into others, conſiſting only of two Terms, one of which compoſed of a certain Number of $x$, either integral or fractional, and the other a Term entirely known; ſuch are the Equations $4x = 8200$, $\frac{2}{3}x = 5320$ and ſolved in the Articles V and VI.

Let us now reſume what we have ſaid concerning thoſe Equations, in order to deduce from thence general Principles applicable to all other Equations.

From the Equation $4x = 8200$ we deduced $x = \frac{8200}{4}$ becauſe $4x$ being equal to 8200, it followed that one $x$ muſt be the fourth Part of that Sum, from this Reaſoning, and from thoſe which may be framed in like Manner for other Numbers of $x$, we deduce this general Principle, that the Multiplier which affects the unknown Quantity in one of the Members of the Equation may be taken away, by making it ſerve as a Diviſor to the other Member. Method of making the multiplicator which affects the unknown quantity diſappear.

XV.

From the Equation $\frac{7}{3}x = 5320$ we deduced $7x = 3 \times 5320$, obſerving that if the third Part of $7x$ was equal to 5320, the whole $7x$ muſt be equal to three Times that Sum. From whence we deduce this general Principle, that the Diviſor which affects the unknown Quantity in a Member of an Equation may be taken away, by making it ſerve as a Multiplier to the other Member. Method of making the diviſor which affects the unknown quantity diſappear.

XVI.

With thoſe Rules we are enabled to ſolve all Kinds of Equations of the firſt Degree. To exerciſe Beginners here are ſome Examples.

$\frac{6}{5}x - 90 + \frac{2}{3}x = \frac{4}{3}x - 82$ is changed by Tranſpoſition into $\frac{6}{5}x + \frac{2}{3}x - \frac{4}{3}x = 90 - 82$, and by Reduction becomes $\frac{6}{5}x - \frac{2}{3}x = 8$, or $\frac{18}{15}x - \frac{10}{15}x = 8$, or $\frac{8}{15}x = 8$, or $8x = 8 \times 15$, or infine $x = 15$.

In like Manner $\frac{2}{7}x + 9 = \frac{1}{3}x - 10$ by Tranſpoſition is changed into $\frac{1}{3}x - \frac{2}{7}x = 10 + 9$, or $\frac{1}{3}x - \frac{2}{7}x = 19$, or $\frac{1}{21}x = 19$, or $x = 399$. Examples of equations of the firſt degree ſolved by the foregoing principles.

Likewiſe $\frac{2}{9}x - 40 - \frac{1}{4}x = 60 - \frac{7}{5}x$, by Tranſpoſition becomes $\frac{2}{9}x - \frac{1}{4}x + \frac{7}{5}x = 100$, which by reducing firſt $\frac{2}{9}$ & $\frac{1}{4}$ to the ſame Denominator, becomes $\frac{7}{5}x - \frac{1}{36}x = 100$, and by reducing $\frac{7}{5}$ & $\frac{1}{36}$ to the ſame Denominator, becomes $\frac{247}{180}x = 100$, or $x = \frac{18000}{247}$.

XVII.

Inſtead of reducing all the Fractions to the ſame Denominator, all the Diviſors of the Equation may be made to vaniſh one after the other, by the following Method, which muſt have ſoon occurred to thoſe who firſt handled thoſe Kinds of Equations.

Let the foregoing Example $\frac{2}{9}x - \frac{1}{4}x + \frac{7}{5}x = 100$ be reſumed, it is manifeſt that if the two Members of this Equation be multiplied by 9, the two Products will be the ſame; for equal Quantities multiplied by the ſame Number ſhould give the ſame Product, we will have by this Multiplication $\frac{18}{9}x - \frac{9}{4}x + \frac{63}{5}x = 900$, which on Account of $\frac{18}{9}x = 2x$, Method of making the fractions of an equation diſappear.

is reduced to $2x - \frac{9}{4}x + \frac{63}{7}x = 900$, in which the Divisor 9 has vanished, and it is easy to perceive that it should necessarily do so; for $\frac{2}{9}$ of any Quantity whatsoever multiplied by 9 should give 2 Integers of this Quantity. To make 4 vanish in the same Manner, I multiply all the Terms of the Equation by 4, only observing that with Respect to the Term $\frac{9}{4}x$, the Multiplication by 4 is performed by taking away the Denominator 4. Hence there results $8x - 9x + \frac{252}{5}x = 3600$, or $\frac{252}{5}x - x = 3600$, which by multiplying the two Members by 5, will become $252x - 5x = 28000$, or $247x = 18000$, or $x = \frac{18000}{247}$.

The general Principle which results from hence, is that to make a Divisor of a Term vanish, we have only to multiply all the other Terms by the Divisor, and take it away from the Term it affects.

XVIII.

Another method whereby they are made to disappear all at once.

We may find another Method for taking away all the Divisors at once, by observing that if we multiply all the Terms by the same Number divisible by each of those Divisors, each Term will be reduced. If we multiply, for Example, the Equation $\frac{2}{9}x - \frac{1}{4}x + \frac{2}{5}x = 100$ by 180, which is divisible by 9, by 4, and by 5, we will have $\frac{360}{9}x - \frac{180}{4}x + \frac{1200}{5}x = 18000$, or $40x - 45x + 252x = 18000$, or $247x = 18000$.

Now to find this Number which may be divisible by all the Divisors, we have only to multiply successively those Divisors into each other. Let the Divisors of the Equation $\frac{7}{3}x + \frac{1}{5}x = 160 - \frac{2}{7}x$ be proposed to be taken away. I first multiply 3 by 5, and afterwards their Product 15 by 7, which gives 105 for the Number divisible by 3, 5, 7; this Number being found, I multiply the whole Equation by it, which gives me $\frac{735}{3}x + \frac{105}{5}x = 16800 - \frac{210}{7}x$, or $245x + 21x = 16800 - 30x$.

To render this Operation more comodious, instead of forming the Product 105 of the three Divisors, it suffices to write it down thus, $3 \times 5 \times 7$, and then we will have $\frac{7 \times 3 \times 5 \times 7}{3}x + \frac{7 \times 3 \times 5}{5}x = 160 \times 3 \times 5 \times 7 - \frac{7 \times 5 \times 3 \times 2}{7}x$, where we see by Inspection, that the Number 3 should vanish in the Numerator of the first Fraction, because the Division by 3 should be destroyed by the Multiplication by 3, it is the same with Respect to 5 and 7, which are at the same Time in the Numerators and Denominators of the other Fractions.

By this Means we arrive at the Equation $7 \times 5 \times 7x + 7 \times 3x = 160 \times 3 \times 5 \times 7 - 2 \times 3 \times 5x$, which after performing the Multiplications indicated by the Signs $\times$, becomes $245x + 21x = 16800 - 30x$ delivered from Fractions.

XIX.

To follow in the most likely Manner possible the Order of the Inventors, we shall dwell no longer on the Method of disengaging the un-

known Quantity, but return to the Manner of reducing Problems to Equations. The Reſolution of Equations might, independent of the Problems to which they relate, employ the firſt Analyſts when this Science had been advanced to a certain Point, but it is to be preſumed that thoſe who layed the Foundations of it, only examined Equations as they related to Problems, theſe being the Concluſions to which Problems were to be brought. Beſides, there ſometimes occur Complications in Equations which had never been thought of, if the Nature of the Problems ſought had not pointed them out.

We cannot explain more fully, the Manner of reducing Problems to Equations, than has been already done, Art. VIII. but will proceed to give ſeveral Examples in order to render this Reſearch familiar to Beginners.

Third Problem.

*To pay a certain Number of Workmen at the Rate of £3 each, £8 is wanting to the Perſon who employed them, but on giving them £2 each he has £3 left, it is required to determine how much Money he had.*

Let $x$ expreſs the Number of Pounds which this Man poſſeſſes, wherefore $x + 8$ will be the Sum ſufficient to pay all the Workmen at the Rate of £3 each; and ſince the Number of Workmen ſhould be three Times leſs than that which is expreſſed by this Sum, it will be expreſſed by the third Part of $x + 8$, which is denoted thus, $\frac{x+8}{3}$; for in ſpecious as in numeral Arithmetick, a Line placed between two Quantities indicates the Diviſion of the ſuperior Quantity by the inferior.

In ſpecious as in numeral arithmetick a bar is employed to indicate diviſion.

Moreover, ſince there Remains £3 after £2 has been given to each Man, $x - 3$ will be the Sum ſufficient to pay all the Workmen at this rate. Wherefore $\frac{x-3}{2}$ will denote the Number of Workmen, but ſince we have two Values of the ſame Number, they muſt be equal, the Problem is therefore reduced to the Reſolution of the Equation $\frac{x-3}{2} = \frac{x+8}{3}$.

To ſolve this Equation, I firſt make the Diviſor 2 of the Member $\frac{x-3}{2}$ diſappear, by multiplying the other Member by this ſame Number 2, which will change the Equation into $x - 3 = \frac{2x+16}{3}$; for it is manifeſt that the Double of $\frac{x-3}{2}$ is $x - 3$, and that the Double of $\frac{x+8}{3}$ will be $\frac{2x+16}{3}$ for the ſame Reaſon that $2x + 16$ is double of $x + 8$. I afterwards make the Diviſor 3 vaniſh in the Equation

$\frac{2x+16}{3} = x-3$, by multiplying the ſecond Member by 3 and taking it away from the firſt, which will give $2x+16 = 3x-9$, or $x = 25$.

If we would know how many Workmen there were, we muſt take one of the two Expreſſions $\frac{x-3}{2}$ or $\frac{x+8}{3}$ found for this Number, $\frac{x-3}{2}$ for Example. Now ſince $x = 25$, $x-3$ will be 22, and conſequently $\frac{x-3}{2}$ will be $\frac{22}{2} = 11$ Number of Workmen ſought.

XX.

It is to be obſerved with Reſpect to the Equation $\frac{x-3}{2} = \frac{x+8}{3}$ that it is not permitted in order to apply the Rule of Art. XI. to change the Side and Sign of the Quantities $-3$ and $+8$, and write the Equation thus $\frac{x-8}{2} = \frac{x+3}{3}$, becauſe the Number $-3$ is not, properly ſpeaking, a Term of the firſt Member, but only a Term of its Dividend $x-3$; the Quantity $\frac{x-3}{2}$ being in Reality only one Term of the Equation, as alſo $\frac{x+8}{3}$. To apply therefore the Rule of Art. XI. we muſt firſt take, as indicated by the Number 2 which is under the firſt Bar, the Half of $x-3$ which will be $\frac{1}{2}x - \frac{3}{2}$; afterwards we muſt take, on Account of 3 which is under the other Bar, the third Part of $x+8$ which will be $\frac{1}{3}x + \frac{8}{3}$: Then equalling thoſe two Quantities we will have the Equation $\frac{1}{2}x - \frac{3}{2} = \frac{1}{3}x + \frac{8}{3}$ in which we may make what Tranſpoſitions we pleaſe.

XXI.

Another ſolution of the ſame problem.

The foregoing Problem may be alſo ſolved after the following Manner.

Let $y$ expreſs the Number of Workmen, $3y$ will be the Sum of Money which ſhould be diſtributed among them at the Rate of £3 each. But £8 is wanting to ſatisfy them at this Price: Wherefore $3y-8$ is the Sum of Money which their Employer poſſeſſes.

On the other Hand $2y$ will be the Sum ſufficient to pay the Workmen at the Rate of £2 each, and in this Caſe there will remain £3 Wherefore $2y+3$ is another Expreſſion of the Sum which their Employer poſſeſſes.

We muſt therefore equal the two Quantities $2y+3$ and $3y-8$, or what amounts to the ſame, we muſt ſolve the Equation $2y+3 = 3y-8$ to obtain the Value of $y$. Which being done by the foregoing Principles, 11 will be found for $y$, that is, for the Number of Workmen required.

XXIII.

*A Meſſenger being departed 9 Hours from a certain Place, travelling at the Rate of 5 Miles in 2 Hours, another Meſſenger is ſent after him, who travels at the Rate of 11 Miles in 3 Hours; it is required to find what Number of Miles the ſecond Meſſenger will ride before he overtakes the firſt.* Fourth problem.

Let $x$ be this Number of Miles, it is manifeſt that this Diſtance ſhould be equal to the Number of Miles which the firſt Meſſenger made during his 9 Hours of Advance, together with what he made whilſt the ſecond Meſſenger is on Rout. I inveſtigate firſt the Number of Miles travelled by the firſt Meſſenger in 9 Hours, by the following Proportion or Rule of Three.

As 2 Hours are to 5 Miles, ſo 9 Hours are to a fourth Term, which, according to the known Rules of Arithmetick, is obtained by multiplying the ſecond Term 5 of the Proportion by the third 9, and dividing their Product by the firſt 2; and which conſequently will be $\frac{45}{2}$ Number of Miles made by the firſt Meſſenger in 9 Hours.

But as the Analyſts always endeavour to expreſs their Operations in the moſt conciſe Manner; they denote this Proportion thus; - - Manner of expreſſing proportions in ſpecious arithmetick

Hours. Miles. Hours. Miles.

$2 : 5 = 9 : \frac{45}{2}$ the Signs: Serving, one to compare 2 to 5 and the other 9 to $\frac{45}{2}$, and the Sign $=$ ſerving to denote the Equality which ſubſiſts between the Ratios of 2 to 5 and that of 9 to $\frac{45}{2}$.

To find afterwards the Diſtance rode by the ſame Meſſenger during the Time the ſecond Meſſenger travels the Number of Miles $x$, I inveſtigate, firſt, the Time which the ſecond Meſſenger takes to travel the Diſtance $x$, by Means of the following Proportion; - - - - - - - -

Miles. Hours. Miles. Hours.

$11 : 3 = x : \frac{3}{11}x$. Whereby, without embarraſſing ourſelves about the Number of Miles expreſſed by $x$, we learn that it ſuffices to multiply this Number by 3 and divide it by 11, to obtain the Number of Hours employed by the ſecond Meſſenger in travelling the Number of Miles $x$.

Now without minding whether the Number of Hours expreſſed by $\frac{3}{11}x$ be known, or unknown, I make the following Proportion;

Hours. Miles. Hours. Miles.

$2 : 5 = \frac{3}{11}x : \frac{15}{22}x$.

The fourth Term of which $\frac{15}{22}x$ expreſſes the Number of Miles travelled by the firſt Meſſenger in the Time $\frac{3}{11}x$, that is, before he has been overtaken.

By this Means we have the ſame Quantity expreſſed two different Ways; for the Diſtance rode by the ſecond Meſſenger is firſt expreſſed by $x$, ſecondly it is the Sum of $\frac{45}{2}$ Miles which the firſt Meſſenger had travelled before the other had ſet out, and of $\frac{15}{22}x$ Miles which this ſame Meſſenger muſt have travelled, before he was overtaken.

Equalling therefore those two Expreſſions we will have the Equation $\frac{45}{7}x = \frac{13}{4}x$ which by the foregoing Rule gives $x = 70 + \frac{2}{3}$.

XXIII.

If the firſt Meſſenger, beſides the Advantage he has of having ſet out ſooner, has alſo that of having ſet out from a Place more advanced, the Problem though more complicated, will be eaſily reduced to the ſame Principles.

Let the firſt Meſſenger, for Example, going to *Londonderry*, ſet out from *Caſhel* on *Monday* at 8 in the Afternoon, travelling at the Rate of 7 Miles in 3 Hours, and the ſecond Meſſenger purſuing the firſt ſet out on *Tueſday* at 10 in the Morning from *Cork*, ſuppoſed to be 34 Miles diſtant from *Caſhel*, travelling at the Rate of 13 Miles in 4 Hours, it is required to find what Number of Miles the ſecond Meſſenger will ride before he overtakes the firſt.

To ſolve this Problem, we muſt take the Difference between 8 in the Afternoon and 10 in the Morning, which is 14 Hours; and as the firſt travels 7 Miles in 3 Hours, we will have by the following Rule of three;

Hours. Miles. Hours. Miles.

3 : 7 = 14 : $\frac{98}{3}$, which being added to 34 Miles of Advance will give $34 + \frac{98}{3}$ or $\frac{200}{3}$ Miles for the Diſtance of the firſt Meſſenger from *Cork*, when the ſecond ſet out. I afterwards make as above the following Proportion;

Miles. Hours. Miles. Hours.

13 : 4 = $x$ : $\frac{4}{13}x$

Number of Hours in which the ſecond Meſſenger travels the Diſtance $x$.

But during this ſame Number of Hours, the firſt Meſſenger would have travelled a Number of Miles with will be found thus

Hours. Miles. Hours. Miles.

3 : 7 = $\frac{4}{13}x$ : $\frac{28}{39}x$, We will have therefore the following Equation, $x = \frac{28}{39}x + \frac{200}{3}$, from whence is deduced, by the Rules explained above $x = 236 + \frac{4}{11}$, Number of Miles travelled by the ſecond Meſſenger before he overtakes the firſt.

XXIV.

As ſoon as the firſt Analyſts had found out the Solution of any intereſting Problem, they did not fail of making ſeveral Applications by varying the Numbers given in thoſe Problems. For Example, they would have repeated ſeveral Times the foregoing Queſtion, by varying the Ratio of the Velocities of the Meſſengers, and the Diſtances of the Places from each other they ſet out from. In thoſe different Applications, they perceived that a Part of the Operation was repeated in each particular Example of the ſame Problem, which might be performed once for all by inveſtigating a Solution which was not reſtrained to any particular Number, but applicable to every Number given. To explain what they imagined in this Reſpect we will reſume the foregoing Problem, and treat it after a general Manner as poſſibly we can.

Let the Diſtance of the Meſſengers from each other be expreſſed by $a$, when the Queſtion is brought to a Concluſion, we may make this Letter repreſent any Number of Miles we pleaſe.

General ſolution of the foregoing problem

Let the Number of Hours which the Departure of the firſt Meſſenger precedes that of the ſecond be expreſſed by the Letter $b$,

Let the Velocity of the firſt Meſſenger be ſuch that he travels the Number of Miles $c$ in the Number of Hours $d$.

Let the Velocity of the ſecond Meſſenger be ſuch that he travels the Number of Miles $e$ in the Number of Hours $f$.

Finally, let the Number of Miles which the ſecond Meſſenger travels before he overtakes the firſt be expreſſed by $x$.

The firſt letters of the alphabet ſerve to expreſs the known quantities, and the laſt letters the unknown ones.

It is uſual in Specious Arithmetick to expreſs the known Quantities by the firſt Letters of the Alphabet $a$, $b$, $c$, &c. and the Quantities ſought by the laſt $s$, $t$, $u$, $x$, &c.

To find now, according to the Method purſued in the foregoing Example, the Number of Miles travelled by the firſt Meſſenger in the Number of Hours $b$, we muſt inveſtigate the fourth Term of a Proportion, of which the firſt Term is the Number of Hours $d$, the ſecond the Number of Miles $c$, the third the Number of Hours $b$, and it is manifeſt that this Operation will be performed, as in all other Rules of three, by multiplying the ſecond and third Term, one by the other, and dividing their Product by the firſt Term.

Letters which follow one another without having any ſign between them are to be conſidered as multiplied by each other.

As to the Manner of expreſſing the Product of thoſe Terms which are no more as heretofore Numbers, but Letters adapted to expreſs any Numbers, what has appeared the moſt ſimple, is to place the Letters to be multiplied, beſide each other; with Reſpect to Diviſion, we have ſeen already that in ſpecious as in numeral Arithmetick, a Bar is placed between the Quantities to be divided.

By this Means the foregoing Proportion is wrote down thus, $d : c = b : \frac{bc}{d}$.

Having therefore $\frac{bc}{d}$ to expreſs the Diſtance which the firſt Meſſenger travelled before the ſecond ſet out, if we add to it the Diſtance $a$ of the Places from whence they ſet out, we will have for the Number of Miles which the firſt Meſſenger is advanced at the Time of the Departure of the ſecond $a + \frac{bc}{d}$.

To find afterwards the Diſtance travelled by the firſt Meſſenger whilſt the other purſues him and rides $x$ Miles, I firſt inveſtigate, as above, the Time in which the ſecond Meſſenger rides the Diſtance $x$, by Means of the Proportion $e : f = x : \frac{fx}{e}$ whoſe firſt Term is the

Number of Miles $e$, the ſecond, the Number of Hours $f$, the third the Number of Miles $x$, and the fourth $\frac{fx}{e}$ the Time ſought.

Now, whatever be the Number of Hours $\frac{fx}{e}$ which the ſecond Meſſenger employed to overtake the firſt, it is obvious if a Proportion be made of which the three firſt Terms are, 1°. the Number of Hours $d$; 2°. the Number of Miles $c$; 3°. the Number $\frac{fx}{e}$, the fourth Term will be the Diſtance travelled by the firſt Meſſenger in the ſame Time that the ſecond travelled the Diſtance $x$.

I write down this Proportion thus $d : c = \frac{fx}{e} : c \times \frac{\frac{fx}{e}}{d}$ Number of Miles travelled by the firſt Meſſenger in the Time the ſecond rides $x$ Miles.

But the Number of Miles travelled by the firſt Meſſenger together with the Number of Miles $a + \frac{bc}{d}$ which he had advance, ſhould be equal to the Number of Miles travelled by the ſecond.

We have therefore the Equation $x = a + \frac{bc}{d} + c \times \frac{\frac{fx}{e}}{d}$. Now recollecting the Operations on Fractions, I find that to multiply a Fraction as $\frac{6}{3}$ by 4, we are to multiply the Numerator by this Number and write down $\frac{6 \times 4}{3}$ or $\frac{24}{3}$. in like Manner to multiply $\frac{fx}{e}$ by $c$ we muſt multiply $c$ by $fx$ which gives $\frac{cfx}{e}$ for $c \times \frac{fx}{e}$. It alſo appears that to divide a Fraction as $\frac{5}{3}$ by any Number as 6, we are to multiply the Denominator 3 by this Number 6, and write down $\frac{5}{3 \times 6}$ or $\frac{5}{18}$. In like Manner to divide the Fraction $\frac{cfx}{e}$ by $d$, we muſt write down $\frac{cfx}{ed}$.

Having thus changed the foregoing Expreſſion $c \times \frac{\frac{fx}{e}}{d}$ into $\frac{cfx}{de}$ the Equation to be ſolved will be $x = a + \frac{bc}{d} + \frac{cfx}{de}$. Operation which requires we ſhould begin as has been taught Art. XVIII. by multiplying all the Terms, except the laſt, by the Diviſor $de$ in order to take it away from this Term.

By this Operation we ſhall have $dex = ade + \frac{bcde}{d} + cfx$,

or $dex = ade + bce + cfx$, becauſe $\frac{bcde}{d}$ is the ſame as $bce$, ſince the Quantity $bce$ remains the ſame when multiplied and divided by $d$.

Carrying the Term $cfx$ over to the firſt Member, we will have $dex - cfx = ade + bce$.

In order to find $x$ in this Equation, I obſerve that if the Numbers $de$ and $cf$ which expreſs how often $x$ is contained in the Terms $dex$ and $cfx$ were known, we would ſubtract the ſecond from the firſt, and the Remainder which would expreſs the Quantity of $x$ contained in the firſt Member of the Equation, would ſerve as a Diviſor to the ſecond Member, to obtain the Value of $x$. But without knowing the Numbers $de$ and $cf$, it is manifeſt that $de - cf$ expreſs their Difference, and conſequently the Quantity of $x$ which is contained in the firſt Member of the Equation $dex - cfx = ade + bce$. Therefore the Value of $x$ will be what reſults by dividing the ſecond Member by this Number $de - cf$. wherefore $x = \frac{ade + bce}{de - cf}$, and this is the general Solution of the foregoing Problem; for when the Quantities $a, b, c, d, e, f$ are known, we have no more to do than to make the Uſe of them indicated by this general Value of $x$, that is, to multiply ſucceſſively $a, d, e$, one by the other: To add to this Product that ariſing from the Multiplication of the Quantities $b, c, e$, one by the other, and to divide the Sum of thoſe Products, by the Number which is the Difference of the Product of $c$ into $f$ compared to the Product of $d$ into $e$, and we will have by this Operation any particular Solution we pleaſe.

XXV.

Example of the foregoing ſolution in numbers.

Let us ſuppoſe, as in Art. XXIII the Diſtance of the two Meſſengers from each other to be 34 Miles, that the firſt Meſſenger ſets out 14 Hours ſooner than the ſecond, and travels 7 Miles in 3 Hours, and that the ſecond rides 13 Miles in 4 Hours, we will have $a = 34$, $b = 14$, $c = 7$, $d = 3$, $e = 13$, $f = 4$, which will give $ade = 34 \times 3 \times 13$, that is, $= 102 \times 13 = 1326$, $bce = 14 \times 7 \times 13 = 1274$, and conſequently $ade + bce = 2600$, $de = 39$, $cf = 28$, and of Courſe $de - cf = 11$.

From whence we deduce $x = \frac{ade + bce}{de - cf} = \frac{2600}{11} = 236 + \frac{4}{11}$, the ſame as found in Art. XXIII.

From this general Solution we may alſo deduce the firſt Caſe calculated in Art. XXII. in which the two Meſſengers were ſuppoſed to ſet out from the ſame Place, the Firſt 9 Hours ſooner than the Latter, and travelling at the Rate of 5 Miles in 2 Hours the ſecond travelling at the Rate of 11 Miles in 3 Hours, in this Caſe $a = 0$, $b = 9$, $c = 5$,

$d = 2$, $e = 11$, $f = 3$, and ſubſtituting thoſe Values in the general Formula or Value of $x$, we will have $x = \frac{9 \times 5 \times 11}{4 \times 11 - 5 \times 3} = \frac{495}{7} = 70 + \frac{5}{7}$, as found in Art. XXII, in the ſame Manner we may make as many Applications as we pleaſe.

XXVI.

No ſooner had the firſt Analyſts found out the Method of rendering the Solution of a Problem general by making Uſe of Letters inſtead of Numbers, but probably they ever after conſidered Problems in their greateſt Generality; it is therefore proper to accuſtom Beginners to treat them in this Manner. With this View we ſhall ſolve the following Problem.

Fifth problem.

*A Workman can do a Piece of Work expreſſed by* a *in a Time expreſſed by* b; *a ſecond can do a Piece of Work* c *in a Time* d; *a third can do a Piece of Work* e *in a Time* f, *it is required to determine in what Time thoſe three Workmen together can do a Piece of Work* g.

Let $x$ be the Time ſought, the Work done by the firſt in this Time will be found by the following Proportion: $b : a = x : \frac{ax}{b}$.

The Work done by the ſecond in the ſame Time will be found by the following Proportion: $d : c = x : \frac{cx}{d}$.

Finally we will have the Work done in the ſame Time by the third Workman, by Means of the following Proportion; $f : e = x : \frac{ex}{f}$.

Wherefore $\frac{fx}{f} + \frac{cx}{d} + \frac{ax}{b}$ is the Work done by the three Workmen together in the Time ſought, but this Work ſhould be equal g, we have therefore the Equation $\frac{ex}{f} + \frac{cx}{d} + \frac{ax}{b} = g$.

To ſolve it, I multiply according to the Rules of Art. XVIII. the whole Equation by the Product $fbd$ of the Diviſors, and I have $\frac{edbfx}{f} + \frac{cdbfx}{d} + \frac{axfdb}{b} = bdfg$, which is reduced to $edbx + fcbx + adfx = bdfg$, in which obſerving that $edb + fcb + adf$ expreſſes the Number of $x$ contained in the ſecond Member, I conclude that $x = \frac{bdfg}{bde + bcf + adf}$.

XXVII.

Example in Numbers.

To make ſome Application of this Problem, let us ſuppoſe that a Maſon can do 7 Feet ſquare of a Wall in 5 Days, that a ſecond Maſon can do 10 Feet in 3 Days, and a third 11 Feet in 4 Days, it is required

in what Time thoſe three Maſons together, would do 150 feet ſquare of the ſame wall.

According to thoſe Suppoſitions we will have $a = 7$; $b = 5$; $c = 10$; $d = 3$; $e = 11$; $f = 4$; $g = 150$; and conſequently $bdfg = 5 \times 3 \times 4 \times 150 = 9000$, $bde = 5 \times 3 \times 11 = 165$; $bcf = 5 \times 10 \times 4 = 200$, $adf = 7 \times 3 \times 4 = 84$; wherefore $x = \frac{9000}{449} = 20 + \frac{20}{449}$ number of Days in which the propoſed Work will be finiſhed.

XXVIII.

Another Example.

Let us ſuppoſe it be required in what Time a Ciſtern of 200 cubick Feet will be filled by three Pipes, the firſt of which can fill 9 cubick Feet in $2\frac{1}{2}$ Days, the ſecond 15 cubick Feet in $3\frac{1}{3}$ Days, and the third 19 cubick Feet in $5\frac{1}{4}$ Days; we will have $a = 9$, $b = 2\frac{1}{2}$, or $\frac{5}{2}$; $c = 15$; $d = 3\frac{1}{3}$, or $\frac{10}{3}$; $e = 19$, $f = 5\frac{1}{4}$, or $\frac{21}{4}$; $g = 200$.

Wherefore, $x = \dfrac{\frac{5}{2} \times \frac{10}{3} \times \frac{21}{4} \times 200}{\frac{5}{2} \times \frac{10}{3} \times 19 + \frac{5}{2} \times 15 \times \frac{21}{4} + 9 \times \frac{10}{3} \times \frac{21}{4}}$

which becomes $\dfrac{\frac{210000}{2 \times 3 \times 4}}{\frac{950}{2 \times 3} + \frac{1575}{2 \times 4} + \frac{1890}{3 \times 4}}$

To reduce this Quantity I multiply the Numerator and Denominator of the firſt Fraction of the Diviſor by 4; the Numerator and Denominator of the ſecond by 3; and the Numerator and Denominator of the third by 2, which gives $\dfrac{\frac{210000}{2 \times 3 \times 4}}{\frac{3800}{2 \times 3 \times 4} + \frac{4725}{2 \times 3 \times 4} + \frac{3780}{2 \times 3 \times 4}}$

or $\dfrac{\frac{210000}{2 \times 3 \times 4}}{\frac{12305}{2 \times 3 \times 4}}$, or $\frac{210000}{12305}$, or $17 + \frac{163}{2461}$. Number of Days ſought in which the three Pipes running at the ſame Time will fill the propoſed Ciſtern.

XXIX.

The application of thoſe rules has produced ſeveral operations of ſpecious arithmetick

It appears from the two foregoing Problems that the Rules delivered in Art. x. &c. for ſolving numeral Equations of the firſt Degree are equally applicable to literal Equations; but it appears at the ſame Time that Beginners require to be directed in the Manner of employing them, we think ourſelf ſo much the more obliged to aſſiſt them by a great Number of thoſe Applications, as it is probable that it is to them we

are indebted for a great Number of uſeful Operations of Specious Arithmetick, which we ſhall diſcover as it were accidentally.

First example of the reſolution of literal equations.

Let it be propoſed to ſolve the Equation $2ac + ab - ax = 3ac + 2dx - 5ab - dx$.

I firſt carry the Terms $3ac$ and $-5ab$ over to the other Member of the Equation having changed their Signs, and there reſults $2ac + ab - ax - 3ac + 5ab = 2ax - dx$. I carry likewiſe the Term $-ax$ over to the other Side, obſerving alſo to change its Sign, which gives me $2ac + ab - 3ac + 5ab = 2ax - dx + ax$. I afterwards reduce this Equation, 1°. by adding $ab$ to $5ab$, which gives me $6ab$; 2°. by ſubſtituting $-ac$ for the Terms $2ac$ and $-3ac$; 3°. by ſubſtituting $3ax$ for $2ax + ax$; the propoſed Equation therefore becomes $6ab - ac = 3ax - dx$ which gives $x = \frac{6ab - ac}{3a - d}$.

XXX.

Second example of the reſolution of literal equations.

Let $5ab + 2ax - 3bd = 2ab - 5ax + 7bd - ac - dx$; the Terms $5ab - 3bd$ will become $-5ab + 3bd$ when carried over to the ſecond Member, and the Terms $-5ax - dx$ will become $+5ax + dx$ when carried over to the firſt; we will have therefore $2ax + 5ax + dx = 2ab + 7bd - ac - 5ab + 3bd$, which is reduced to $7ax + dx = 10bd - 3ab - ac$ by ſubſtituting $7ax$ for $2ax + 5ax$, $10bd$ for $7bd + 3bd$, and $-3ab$ for $2ab - 5ab$. Now diſengageing $x$ in this Equation we will have $x = \frac{10bd - ac - 3ab}{7a + d}$.

XXXI.

Reduction of quantities to their moſt ſimple expreſſion.

In the Reſolution of the two foregoing Equations we had Occaſion to reduce to a more ſimple Expreſſion different Terms of the ſame Kind, ſuch as $2ac$ and $-3ac$; $5ab$ and $ab$, &c. As this Operation is frequently neceſſary in Equations which are to be ſolved and in the other Parts of Specious Arithmetick, Beginners ſhould render it familiar to themſelves, to aſſiſt them thereto; Here follow ſome Examples.

Let $15abc - 13bcd - 7abc + 29bcd - 5abf + 9abc + 6cbi$ be propoſed to be reduced.

Affirmative terms are thoſe which are preceded by $+$. Negative thoſe which are preceded by $-$.

I firſt take the Terms $15abc$, $-7abc$ and $9abc$ which are of the ſame Kind, I add the two Terms $15abc$ and $9abc$, which are both one and the other Affirmative, that is, affected with the Sign $+$; I afterwards ſubtract from their Sum which is $24abc$, the Term $7abc$ becauſe it is negatif or preceded by the Sign $-$, by this Means the three Terms $15abc - 7abc + 9abc$ are reduced to $17abc$. In like Manner, inſtead of $29bcd - 13bcd$ I ſubſtitute $16bcd$. As to the Terms $-5abf$ and $6cbi$ which are alone of their Kinds, I write them down as they are, hence the Quantity reduced will be $17abc + 16bcd - 5abf + 6cbi$.

The Quantity $\frac{2}{3}ab - \frac{1}{2}ac + \frac{1}{4}ax - ad + \frac{1}{7}ab + \frac{1}{3}ax$, will be reduced to $\frac{23}{21}ab + \frac{7}{12}ax - \frac{1}{2}ac - ad$.

The Quantity $2acd - 5acb - 3acd + 3acb - 6bfi$ will become after being reduced $-acd - 2acb - 6bfi$, which being entirely negative, shews that the Quantity which was to be reduced includes more negative than affirmative Parts.

XXXII.

The algebraick addition is the same operation as the foregoing.

It is proper to observe here that the Reduction which has been employed in the foregoing Examples, is precisely the same Rule as that which is called Addition; for when two Quantities are proposed to be added, it suffices to write them down one after the other, and afterwards to reduce them to their most simple Expression: If it was required, for Example, to add the Quantity $6ab - 2ac - 3ad$ to $3ab + ac - 2ad + bf$, there is no more to be done than to reduce the Quantity $6ab - 2ac - 3ad + 3ab + ac - 2ad + bf$, which will consequently give $9ab - ac - 5ad + bf$ for the Sum of the two proposed Quantities.

If it was proposed to add the two Quantities $2ac - 3ad + af$ and $ad - 5ac - 2af$, no more is required than to reduce the Quantity $2ac - 3ad + af + ad - 5ac - 2af$. The Reduction being performed there will result $-3ac - 2ad - af$. It may appear at first View surprizing that the Result of an Addition should be a negative Quantity; but this Difficulty will soon be solved when it is observed that the two Quantities $2ac - 3ad + af$ and $ad - 5ac - 2af$ must be both Negative, or at least one of the two negative and greater than the other.

This will more readily appear by some Examples in Numbers. Let us first suppose $a = 2$, $c = 3$, $d = 4$; $f = 5$, in this Case, instead of $2ac - 3ad + af$ we will have $12 - 24 + 10$ or simply $-2$, and instead of $ad - 5ac - 2af$ we will have $8 - 30 - 20 = -42$ and their Sum will be $-44$, and it is no way surprizing that the Sum of two negatif Quantities should be negatif.

Let us now suppose $a = 6$, $c = 5$, $d = 3$, $f = 2$, we will have $2ac - 3ad + af = 18$ and $ad - 5ac - 2af = -126$. Wherefore the second Quantity being negative, and greater than the first the Sum should be negative.

XXXIII.

In what sense a negative quantity can be said to be added.

It may perhaps be asked, if a negative Quantity can be added to an affirmative one, or rather can a negative Quantity be said to be added. To which I reply that this Expression is exact when the Words to add and to augment are not confounded. Let two Men, for Example, join their Fortunes, be them what they will, I would say they add their Fortunes; let one have Debts and real Effects, if his Debts exceed his Ef-

fects, he will poſſeſs but a negative Fortune, and the Junction of his Fortune with that of the firſt would diminiſh that of the firſt; ſo that the Sum will be found, either leſs than what the firſt poſſeſſes, or even entirely negative.

XXXIV.

From the foregoing operation the algebraick ſubtraction is deduced.

The Reduction taught in the foregoing Articles, has given riſe to another Rule of Specious Arithmetick namely Subtraction; for when, for Example, in the Equation $2ac + ab - ax = 3ac + 2ax - 5ab - dx$ (Art. XXIX.) we carried over the Terms $3ac - 5ab$ to the other Side, after having changed their Signs, and we arrived at the Equation $2ac + ab - ax - 3ac + 5ab = 2ax - dx$, or $-ac + 6ab - ax = 2ax - dx$, I ſay that the Quantity $3ac - 5ab$ was ſubtracted from the Quantity $2ac + ab - ax$ and the Remainder is $-ac + 6ab - ax$. For by making $3ac - 5ab$ diſappear in the ſecond Member of the Equation, a Subtraction of this Quantity was performed, but to preſerve the Equality, a ſimilar Subtraction muſt have been made in the firſt Member; therefore $2ac + ab - ax - 3ac + 5ab$ or $-ac + 6ab - ax$ is what remains of the Quantity $2ac + ab - ax$ after $3ac - 5ab$ has been ſubtracted from it.

Proceſs of ſubtraction.

Hence when there are two Quantities of which one is to be ſubtracted from the other, we muſt change the Signs of the Quantity to be ſubtracted, and write it down after the other, then reduce the Quantities of the ſame Kind, which independant of what has been ſaid may be demonſtrated after the following Manner.

Let $2ac + ab - ax$ be the Quantity from which it is propoſed to ſubtract the Quantity $3ac - 5ab$. It is manifeſt that if $3ac$ was only to be ſubtracted from it, we ſhould write down $2ac + ab - ax - 3ac$, but by ſubtracting the Quantity $3ac$ inſtead of $3ac - 5ab$ a Quantity too great by $5ab$ has been ſubtracted: Wherefore we muſt add the $5ab$ which has been taken away too much by ſubtracting $3ac$. Therefore we muſt write down $2ac + ab - ax - 3ac + 5ab$ for the Remainder of $2ac + ab - ax$ after $3ac - 5ab$ has been ſubtracted from it.

To exerciſe Beginners in this Rule which it is obvious ſhould frequently occur I have added the following Examples.

If from $5ab + 10fg - 3ac + 2de$ we ſubtract $2ab - 5fg + 6ac + de$, there will remain $5ab + 10fg - 3ac + 2de - 2ab + 5fg - 6ac - de$, or $3ab + 15fg - 9ac + de$.

If from the Quantity $6aeb + 3agb - 10bcd$ we ſubtract $abc - 10aeb - 8agb$, there will remain $16aeb - abc - 10bcd + 11agb$.

If from the Quantity $3ac + ab + be$ we ſubtract $-ac - 3ab$, there will remain $4ac + 4ab + be$.

XXXV.

If it appears surprizing that in this subtraction the Remainder $4ac + 4ab + be$ is greater than the Quantity $3ac + ab + be$ from which it was proposed to subtract $-ac - 3ab$, it can only arise from not making a proper Distinction between subtracting and diminishing; for if we consider that to subtract any Quantity, $a$, for Example, from another $b$, is to determine how much $b$ exceeds $a$, it will appear very possible that a Quantity may be increased by a Subtraction. Let it be required to determine, for Example, how much one Man is richer than another, if the latter has only Debts, it is manifest that the Excess of the Riches of the first will be what he possesses more a Sum equal to the Debts of the other.

A quantity is increased when a negative quantity is subtracted from it.

XXXVI.

Let it be proposed to solve the Equation $\frac{cx}{2a} - \frac{ac}{2b} = x - \frac{4ad}{3c}$, to make first the Divisor $2a$ disappear, I make it serve according to Art. XV. as a Multiplier to all the Terms of the Equation, and we will have $cx - \frac{ac \times 2a}{2b} = 2a \times x - \frac{4ad \times 2a}{3c}$, but it is manifest that instead of $ac \times 2a$ we may substitute $2aac$, because the Product of $2a$ into $ac$ should be double of the Product of $a$ into $ac$, and the Product of $a$ into $ac$ is $aac$. Likewise $2a \times x$ will be $2ax$ and $4ad \times 2a$ will be $8aad$; for the Product of $ad$ into $a$ is $aad$ and the Product of $4ad$ into $2a$ should be octuple of that of $ad$ into $a$. The Equation is therefore changed into $cx - \frac{2aac}{2b} = 2ax - \frac{8aad}{3c}$ or $cx - \frac{aac}{b} = 2ax - \frac{8aad}{3c}$, because $\frac{2aac}{2b}$ or $\frac{aac}{b}$ are one and the same. Multiplying the Terms of this Equation by $b$, it will become $b \times cx - aac = 2ax \times b - \frac{8aad}{3c} \times b$ or $bcx - aac = 2abx - \frac{8aabd}{3c}$ which will be again changed into $cbx \times 3c - aac \times 3c = 2abx \times 3c - 8aabd$, or $3bccx - 3aacc = 6abcx - 8aabd$, for the Products of $c$ into $cbx$, $aac$ and $2abx$, will be $bccx$, $aacc$, $2abcx$, and consequently the Products of $3c$ by the same Quantities should be triple of the foregoing, that is, $3bccx$, $3aacc$, $6abcx$; now transposing we will have $3bccx - 6abcx = 3aacc - 8aabd$ which at Length gives $x = \frac{3aacc - 8aabd}{3bcc - 6abc}$.

Third example of the resolution of literal equations.

XXXVII.

In the foregoing Example, the Multiplication of some Quantities

which contained the same Letters, produced the Repetition of those Letters in the Products: Now as the Analysts always endeavour to express themselves in the most concise Manner, instead of repeating a Letter several Times one after another they write it down but once, placing at the Top of it on the right Hand a Figure denoting the Number of Times that this Letter should be repeated. Hence, instead of the foregoing Expression $x = \frac{3\,a\,a\,c\,c - 8\,a\,a\,b\,d}{3\,b\,c\,c - 6\,a\,b\,c}$ they write $x = \frac{3\,a^2\,c^2 - 8\,a^2\,b\,d}{3\,b\,c^2 - 6\,a\,b\,c}$.

A number placed at the top of a letter to the right denotes how often it is repeated by multiplication, and in this case the letter is said to be raised to the power expressed by this number which is called exponent.

When in an Operation the Analyst has Occasion for $a\,a\,a$, that is, for the Product of $a\,a$ by $a$ or for $a$ multiplied by itself twice, he puts down simply $a^3$. In like Manner for $c\,c\,c\,c$, he writes $c^4$. When a Letter is thus repeated or rather considered as repeated by Means of a Number, it is said to be raised to the Power expressed by this Number, and that this Number is its Exponent, thus $c^4$ or $c\,c\,c\,c$ which is the Product of $c$ multiplied three Times by itself is said to be raised to the fourth Power, and 4 is its Exponent. Care is to be taken not to confound the Numbers which serve as Exponents with those which are on the left Hand of the Letters and on the same Line, which are called Coefficients; in $4\,a^2\,c$, for Example, 4 is the Coefficient of the Term, 2 is the Exponent of $a$.

The numbers which are to the left and on the same line are called coefficients.

## XXXVIII.

Let the Equation $\frac{2\,a\,b^2\,x}{3\,c^2\,d} + \frac{5\,a\,c^2}{b^2} = \frac{6\,c\,d^2}{a^2} - 3\,x$, be proposed to be solved, multiplying all its Terms by the Divisor $3\,c^2\,d$, we will have $2\,a\,b^2\,x + \frac{5\,a\,c^2 \times 3\,c^2\,d}{b^2} = \frac{6\,c\,d^2 \times 3\,c^2\,d}{a^2} - 3\,x \times 3\,c^2\,d$.

Fourth example of the resolution of literal equations.

To perform the Multiplications indicated by the Signs $\times$, I observe first that $a\,c^2$ multiplied by $c^2\,d$ should give for Product $a\,c^4\,d$, for if instead of $a\,c^2$ and of $c^2\,d$ we write $a\,c\,c$ and $c\,c\,d$, as may be done, their Product will be $a\,c\,c\,c\,c\,d$, that is, according to the foregoing Art. $a\,c^4\,d$. Having therefore $a\,c^4\,d$ for the Product of $a\,c^2$ into $c^2\,d$, it is manifest that $15\,a\,c^4\,d$ will be the Product of $5\,a\,c^2$ into $3\,c^2\,d$.

In like Manner, the Product of $6\,c\,d^2$ into $3\,c^2\,d$ will be found to be $18\,c^3\,d^3$ and the Product of $3\,x$ into $3\,c^2\,d$ to be $9\,c^2\,d\,x$. Wherefore the foregoing Equation will be changed into $2\,a\,b^2\,x + \frac{15\,a\,c^4\,d}{b^2} = \frac{18\,c^3\,d^3}{a^2} - 9\,c^2\,d\,x$.

Multiplying afterwards this new Equation by $b^2$ it will become $2\,a\,b^4\,x + 15\,a\,c^4\,d = \frac{18\,b^2\,c^3\,d^3}{a^2} - 9\,b^2\,c^2\,d\,x$, and this last by $a^2$, there will result $2\,a^3\,b^4\,x + 15\,a^3\,c^4\,d = 18\,b^2\,c^3\,d^3 - 9\,b^2\,a^2\,c^2\,d\,x$,

and by tranſpoſition $2a^3b^4x + 9b^2a^2c^2dx = 18b^2c^3d^3 - 15a^3c^4d$, from whence at length is deduced $x = \frac{18b^2c^3d^3 - 15a^3c^4d}{2a^3b^4 + 9a^2b^2c^2d}$.

XXXIX.

In the foregoing Examples, we had Occaſion to multiply Quantities expreſſed by a ſimple Term ſuch as $4ad$, $9c^2d$, &c. which are commonly called incomplex Quantities or Monomes, and we found at the ſame Time how this Operation was to be performed. The general Method which reſults from the Reaſoning employed in thoſe particular Examples, is firſt to multiply the Coeficients; to add afterwards the Exponents of the ſame Letters, and to write thoſe which are different one after another. Hence according to this Rule, $3a^4b^3d \times 7a^2bd^2 = 21a^6b^4d^3$; $\frac{2}{3}a^2cd \times \frac{3}{5}ac^3bd = \frac{18}{15}a^3c^4bd^2 = \frac{6}{5}a^3c^4bd^2$; $\frac{2}{3}ac^2de \times 9a^4fg = 6a^5c^2defg$.

Incomplex quantities are thoſe which conſiſt only of one term. Their multiplication deduced from the foregoing examples.

XL.

Let it be propoſed to ſolve the Equation $\frac{a^2c}{2b^2} + \frac{4cx}{3a} = \frac{5ab}{c} + 3a$; multiplying it by $2b^2$ I have $a^2c + \frac{8b^2cx}{3a} = \frac{10ab^3}{c} + 6ab^2$, then multiplying by $3a$, I have $3a^3c + 8b^2cx = \frac{30a^2b^3}{c} + 18a^2b^2$, and performing again the ſame Operation to make the Diviſor $c$ vaniſh, there reſults $3a^3c^2 + 8b^2c^2x = 30a^2b^3 - 18a^2b^2c$, from whence is deduced $x = \frac{30a^2b^3 - 18a^2b^2c - 3a^3c^2}{8b^2c^2}$, which may alſo be wrote thus $x = \frac{30a^2b^3}{8b^2c^2} - \frac{18a^2b^2c}{8b^2c^2} - \frac{3a^3c^2}{8b^2c^2}$; becauſe $8b^2c^2$ dividing the whole Quantity $30a^2b^3 - 18a^2b^2c - 3a^3c^2$ divides each of its Parts.

Fifth example of the reſolution of literal equations.

Now the Value of $x$ thus wrote may be more ſimply expreſſed by reducing each Term. For 1°. inſtead of $\frac{30a^2b^3}{8b^2c^2}$ may be put down $\frac{15a^2b}{4c^2}$, becauſe the Numerator may be conſidered, as the Product of $2b^2$ into $15a^2b$, and the Denominator as the Product of the ſame Quantity $2b^2$ into $4c^2$; dividing therefore both one and the other by the ſame Quantity $2b^2$, there reſults $\frac{15a^2b}{4c^2}$; 2°. inſtead of $\frac{18a^2b^2c}{8b^2c^2}$ we may put down $\frac{9a^2}{4c}$; for the Numerator is the Product of $2b^2c$ into $9a^2$, and the Denominator is the Product of the ſame Quantity $2b^2c$ into $4c$. Inſtead of $\frac{3a^3c^2}{8b^2c^2}$ we may put down $\frac{3a^3}{8b^2}$. Therefore the

Value of $x$ reduced is $\frac{15\,a^2\,b}{4\,c^2} - \frac{9\,a^2}{4\,c} - \frac{3\,a^3}{8\,b^2}$.

XLI.

Diviſion of incomplex quantities deduced from this example.

The Method to be purſued in general in all Operations of the ſame Nature as the foregoing, that is, in Diviſions of incomplex Quantities is eaſily deduced from what has been ſaid, particularly after having ſeen the Multiplication of incomplex Quantities. This Method may be expreſſed thus.

Divide firſt the Coeficients if the Diviſion be poſſible, take away the Letters which have the ſame Exponents in the Numerators and Denominators, divide afterwards the Letters which have different Exponents in the Denominator and Numerator, by ſubtracting the leaſt Exponents from the greateſt, and writing the Remainders in the Place of the greateſt Exponents. As to different Letters there is no more to be done than to copy them.

As this Operation frequently occurs, to render it familiar to Beginners here follow ſome Examples.

$$\frac{9\,a^5\,d^2\,b^2}{3\,a^4\,c^2\,d^2} = \frac{3\,a\,b^2}{c^2}, \quad \frac{18\,a^4\,b\,c\,d}{14\,a\,b^2} = \frac{9\,a^3\,c\,d}{7\,b}, \quad \frac{27\,a^3\,b^2\,c^5}{3\,a^2\,b\,c^2}$$

$$= 9\,a\,b\,c^3, \quad \frac{5\,a^2\,b^4\,c^2}{15\,a\,b^3} = \frac{a\,b\,c^2}{3}.$$

XLII.

Sixth example of the reſolution of literal equations.

Let it be propoſed to ſolve the Equation $\frac{a^2\,x}{b-c} + d\,c = b\,x - a\,c$.

Uſe of Parentheſes in ſpecious arithmetick.

To make the Diviſor $b - c$ vaniſh, I multiply all the Terms by this Diviſor, which gives $a^2\,x + (b-c) \times d\,c = (b\,x - a\,c) \times (b - c)$, where I obſerved 1°. to put $b - c$ in the firſt Member between Parentheſes leſt it might be imagined that only $c$ was to be multiplied by $d\,c$; 2° to put $b\,x - a\,c$ and $b - c$ in the ſecond Member between Parentheſes in Order that it may appear that thoſe two entire Quantities are to be multiplied by each other.

It is now queſtion to perform the Multiplications indicated by the Signs $\times$. Let it be propoſed firſt to multiply $d\,c$ by $b - c$, it is manifeſt that we muſt multiply $d\,c$ by $b$ and from their Product ſubtract the Product of $d\,c$ into $c$; for $b - c$ being leſs than $b$ by the Quantity $c$, the Product of $b - c$ into $d\,c$ ſhould be leſs than that of $b$ into $d\,c$, by the Quantity $c \times d\,c$. Therefore the Product of $b - c$ into $d\,c$ is $b\,d\,c - c\,c\,d$.

To find the Product of $b\,x - a\,c$ into $b - c$; I obſerve that if the two Terms $b\,x - a\,c$ be conſidered as a ſingle Quantity, the Product of this Quantity into $b - c$ ſhould be, the Quantity which the Product of $b\,x - a\,c$ into $b$ exceeds the Product of $b\,x - a\,c$ into $c$. The Queſ-

tion is therefore reduced to two Multiplications ſimilar to the foregoing one and to a Subtraction.

The firſt of thoſe two Multiplications, that of $bx - ac$ by $b$, will give $b^2x - abc$; the ſecond that of $bx - ac$ by $c$, will give $bcx - ac^2$. It remains therefore to ſubtract this laſt Quantity from the firſt, which will give, according to Art. XXXIV. $b^2x - abc - bcx + ac^2$, and this is the Product of $bx - ac$ into $b - c$.

So that the Equation $\frac{a^2x}{b-c} + cd = bx - ac$, or $a^2x + (b-c) \times cd = (bx - ac) \times (b - c)$ is become $a^2x + bcd - c^2d = b^2x - abc - bcx + ac^2$, which, by the ordinary Tranſpoſitions, will give $bcd - c^2d + abc - ac^2 = b^2x - bcx - a^2x$, and at length $x = \frac{bcd - c^2d + abc - ac^2}{b^2 - bc - a^2}$.

XLIII.

Multiplication of compound quantities or multinomials deduced from the foregoing article.

In this Example, we had Occaſion to form a Rule of ſpecious Arithmetick, which we had not as yet employed, and which on ſeveral Occaſions may be of Uſe. This Rule is called the Multiplication of Multinomials. A Multinomial or compound Quantity ſignifies in general a Quantity compoſed of ſeveral Terms. When the Number of Terms of a Quantity are ſpecified, if it conſiſts of two it is called a Binomial; if it conſiſts of three it is called a Trinomial, &c.

Example of the multiplication of multinomials.

To exerciſe Beginners in the Multiplication of thoſe Kinds of Quantities, here follow ſome Examples: Let firſt the Product of $2a^3c^2 - 5a^4b + 6a^5$ into $3ab^2 - 4bcd$ be required.

By reaſoning in the ſame Manner as in the foregoing Article, it will appear, that ſince the Quantity $3ab^2 - 4bcd$ is leſs than $3ab^2$ by $4bcd$, the Product of this Quantity into $2a^3c^2 - 5a^4b + 6a^5$ ſhould be leſs than the Product of $3ab^2$ into $2a^3c^2 - 5a^4b + 6a^5$ by the Product of $4bcd$ into $2a^3c^2 - 5a^4b + 6a^5$. In Conſequence, I write the Product thus $(2a^3c^2 - 5a^4b + 6a^5) \times 3ab^2 - (2a^3c^2 - 5a^4b + 6a^5) \times 4bcd$.

Performing now the two Multiplications indicated by the Signs $\times$, after the ſame Manner as thoſe of ſimple Quantities, we will have $6a^4b^2c^2 - 15a^5b^3 + 18a^6b^2$ for the firſt Product $(2a^3c^2 - 5a^4b + 6a^5) \times 3ab^2$. In like Manner we will have $8a^3bc^3d - 20a^4b^2cd + 24a^5bcd$ for the ſecond Product $(2a^3c^2 - 5a^4b + 6a^5) \times 4bcd$. Then ſubtracting the ſecond from the firſt, as indicated in the foregoing Expreſſion, we will have $6a^4b^2c^2 - 15a^5b^3 + 18a^6b^2 - 8a^3bc^3d + 20a^4b^2cd - 24a^5bcd$ for the Product of the two propoſed Quantities.

XLIV.

If the Multiplier of the foregoing Quantity, contained beſides the two Terms $3ab^2 - 4bcd$, another $- 5abc$, for Example, it is manifeſt that to obtain the whole Product, we muſt ſubtract from the foregoing Quantity, the Product of $2a^3c^2 - 5a^4b + 6a^5$ into $5abc$. For the Multiplier $3ab^2 - 4bcd - 5abc$ being leſs than the Multiplier $3ab^2 - 4bcd$, by $5abc$, the Product of it into $2a^3c^2 - 5a^4b + 6a^5$ ſhould be leſs than the Product of $3ab^2 - 4bcd$ into $2a^3c^2 - 5a^4b + 6a^5$ by $5abc \times (2a^3c^2 - 5a^4b + 6a^5)$. In like Manner if there was another Term in the Multiplier, $3ac^2$ for Example, with the Sign $+$ the Product $3ac^2 \times (2a^3c^2 - 5a^4b + 6a^5)$ ſhould be added to the foregoing Products.

Fundamental principle of multiplication.

It appears in general that any Quantity being given to be multiplied, with the Quantity which is to ſerve as Multiplier, we are to form all the Products of the Multiplicand into each of the Terms of the Multiplier, and add or ſubtract thoſe Products, according as the Terms of the Multiplier are affected with the Sign $+$ or $-$.

To perform this Operation after a regular Manner, proceed thus.

XLV.

Method to be purſued in multiplication.

First write down the Multiplier under the Multiplicand, and draw a Bar under the Multiplier. To form afterwards the firſt Line of the Product which is to be ſet down under this Bar, multiply the firſt Term of the Multiplier by each of the Terms of the Multiplicand, obſerving to prefix to each of thoſe Products, the Sign of the Term of the Multiplicand, if the firſt Term of the Multiplier has no Sign, and conſequently is conſidered as having the Sign $+$.

To form afterwards the ſecond Line which ſhould be wrote under the firſt, multiply the ſecond Term of the Multiplier by all the Terms of the Multiplicand, and if this ſecond Term of the Multiplier has alſo the Sign $+$, the Operation is abſolutely the ſame as for the firſt Line, but if it has the Sign $-$, to each of the Products which compoſe this Line, prefix a Sign contrary to that which affects the Term of the Multiplicand to which it relates. All the other Lines of the Product being formed after the ſame Manner, by Means of the other Terms of the Multiplier multiplied by all thoſe of the Multiplicand, draw a Bar, and add or reduce all thoſe particular Products; the Quantity which reſults will be the Product required.

We ſuppoſed that the firſt Term of the Multiplier had the Sign $+$, if however it had the Sign $-$, it is eaſy to perceive that with Reſpect to this Term, as with Reſpect to the others affected by the Sign $-$, Signs contrary to thoſe of the Terms of the Multiplicand, are to be prefixed to the Product of thoſe Terms.

XLVI.

$$\begin{array}{lll} 2ab - 4ac + ad & & \text{Mult.} \\ 3ab - 5ac + 2ad & & \text{by} \\ \hline \left\{\begin{array}{lll} 6a^2b^2 - 12a^2bc + 3a^2bd & & \\ -10a^2bc + 20a^2c^2 - 5a^2cd & & \text{Product.} \\ +4a^2bd - 8a^2cd + 2a^2d^2 & & \end{array}\right\} & & \\ \hline 6a^2b^2 - 22a^2bc + 7a^2bd + 20a^2c^2 - 13a^2cd + 2a^2d^2 & & \text{Sum.} \end{array}$$

To explain this Method, we shall apply it to an Example: Let it be proposed to multiply the two Quantities $2ab - 4ac + ad$ and $3ab - 5ac + 2ad$. The first being chosen for Multiplicand, and the second for Multiplier, I write down this last under the other, as above.

Application of the foregoing method to an example.

Which being done, I remark that the first Term of the Multiplier is affirmative, and consequently all the Signs of the Terms of the first Line of the Product should be the same as those of the Multiplicand. In Consequence of this Remark, I write down $6a^2b^2$ the Product of $3ab$ into $2ab$, without prefixing any Sign to it, which comes to the same as if it had been affected by the Sign $+$.

I afterwards put down $-$ for the Sign of the second Term of the same Line, because it is the Sign of the second Term of the Multiplicand, and after this $-$ I write $12a^2bc$ Product of $4ac$ into $3ab$. I prefix in like Manner the Sign $+$ of the third Term of the Multiplicand to the third Term of the first Line of the Product, which is $3a^2bd$ Product of $ad$ into $3ab$. The first Line of the Product being thus finished, I remark that the second Term of the Multiplier has the Sign $-$, and consequently that the Signs of the Multiplicand must be changed to form the Terms of the second Line of the Product. Wherefore the first Term of this second Line should have the Sign $-$, which I prefix to the Product $10a^2bc$ of the two Terms $2ab$, $5ac$.

The second Term of the same Line should have the Sign $+$, since the second Term of the Multiplicand has the Sign $-$, I therefore prefix the Sign $+$ to the Product $20a^2c^2$ of the two Terms $4ac$, $5ac$.

The third Term $ad$ of the Multiplicand being preceded by the Sign $+$, the third Term of the second Line will be affected by the Sign $-$ which I prefix to the Product $5a^2cd$ of the two Terms $ad$, $5ac$.

As to the third Line of the Product sought, because the third Term of the Multiplier has the Sign $+$, all the Signs of the Multiplicand are to be preserved, consequently the first Term, that is the Product of $2ab$ into $2ad$ will be $4a^2bd$ preceded by the Sign $+$, the second, that is, the Product of $4ac$ into $2ad$ will be $8a^2cd$ preceded by the Sign $-$, and the third, that is, the Product of $2ad$ into $ad$ will be $2a^2d^2$ preceded by the Sign $+$.

To render this Operation familiar to Beginners, here follow more Examples.

Multiplicand. $5a^3b - 2ab^3 + 4a^2c^2$

Multiplier. $2a^3b - ab^3 + 3a^2c^2$

Product.
$$\left\{\begin{array}{l} 10a^6b^2 - 4a^4b^4 + 8a^5bc^2 \\ -5a^4b^4 + 2a^2b^6 - 4a^3b^3c^2 \\ +15a^5bc^2 - 6a^3b^3c^2 + 12a^4c^4 \end{array}\right\}$$

Sum. $10a^6b^2 - 9a^4b^4 + 23a^5bc^2 + 2a^2b^6 - 10a^3b^3c^2 + 12a^4c^4$

Multiplicand. $2a^3x^2 - 3b^3y^2$ $\qquad$ $5ab + 3ac - c^2$

Multiplier. $2a^3x^2 + 3b^3y^2$ $\qquad$ $-5ab + 3ac - c^2$

Product.
$$\left\{\begin{array}{l} 4a^6x^4 - 6a^3b^3y^2x^2 \\ +6a^3b^3x^2y^2 - 9b^6y^4 \end{array}\right\} \qquad \left\{\begin{array}{l} -25a^2b^2 - 15a^2bc + 5ab^2c \\ +15a^2bc + 9a^2c^2 - 3ac^3 \\ -5ab^2c - 3ac^3 + c^4 \end{array}\right\}$$

Sum. $4a^6x^4 \bullet - 9b^6y^4$ $\qquad$ $-25a^2b^2 + 9a^2c^2 - 6ac^3 + c^4$

Multiplicand. $yy + 2ay - \frac{1}{2}aa$ $\qquad$ $xx - ax$

Multiplier. $yy - 2ay + aa$ $\qquad$ $x + a$

Product.
$$\left\{\begin{array}{l} aayy + 2a^3y - \frac{1}{2}a^4 \\ -2ay^3 - 4aayy + a^3y \\ y^4 + 2ay^3 - \frac{1}{2}aayy \end{array}\right\} \qquad \left\{\begin{array}{l} x^3 - axx \\ +axx - aax \end{array}\right\}$$

Sum. $y^4 \bullet - 3\frac{1}{2}aayy + 3a^3y - \frac{1}{2}a^4$ $\qquad$ $x^3 \bullet - aax$

Multiplicand. $aa + 2ac - bc$ $\qquad$ $2a - 4b$

Multiplier. $a - b$ $\qquad$ $2a + 4b$

Product.
$$\left\{\begin{array}{l} -aab - 2abc + bbc \\ a^3 + 2aac - abc \end{array}\right\} \qquad \left\{\begin{array}{l} 4aa - 8ab \\ +8ab - 16bb \end{array}\right\}$$

Sum. $a^3 + 2aac - aab - 3abc + bbc$ $\qquad$ $4aa \bullet - 16bb$

Multiplicand. $2abx - bxy + a^2x + 3a^2y$

Multiplier. $2ax - 3ay$

Product.
$$\left\{\begin{array}{l} 4a^2bx^2 - 2ab x^2y + 2a^3x^2 + 6a^3xy \\ -6a^2bxy + 3abxy^2 - 3a^3xy - 9a^3y^2 \end{array}\right\}$$

$4a^2bx^2 - 2abx^2y + 2a^3x^2 + 6a^3xy - 6a^2bxy + 3abxy^2 - 3a^3xy - 9a^3y^2$

XLVII.

Let it be propoſed to ſolve the Equation $\frac{ab^2 + abd - abx}{d - c} = ax - ac$, I firſt make the Diviſor $d - c$ diſappear by multiplying $ax - ac$ by $d - c$; which gives $ab^2 + abd - abx = (ax - ac) \times (d - c)$, or $ab^2 + abd - abx = adx - acd - acx + ac^2$, which, by carrying over all the Terms affected by $x$ to one ſide, and the known ones to the other, will become $ab^2 + abd + acd - ac^2 = abx + adx - acx$, from whence is deduced - - - - - - - - $x = \frac{ab^2 + abd + acd - ac^2}{ab + ad - ac}$.

Sixth example of the reſolution of literal equations.

A certain Relation which is perceived to ſubſiſt between the Terms of the Dividend and Diviſor in this Expreſſion, might induce us to think that the Diviſion may be performed exactly, and conſequently invites us to attempt this Operation, which ſhould appear eaſy to execute, after having ſeen that of Multiplication of which it is the Converſe.

To diſcover in effect if $ab + ad - ac$ can divide exactly $ab^2 + abd + acd - ac^2$. I divide firſt one of the Terms of this laſt Quantity by one of thoſe of the firſt, for Example, $ab^2$ by $ab$, and write down the Quotient $b$, I afterwards multiply this Quotient $b$, or rather this firſt Part of the Quotient ſought, by the whole Diviſor $ab + ad - ac$, and ſubtract the Product $ab^2 + abd + abc$ from the Dividend, the Remainder $ab^2 + abd + acd - ac^2 - ab^2 - abd + abc$, or $acd - ac^2 + abc$, is ſtill to be divided by the ſame Diviſor, and its Quotient added to the foregoing $b$ to form the whole Quotient ſought.

Manner of performing the diviſion indicated in this example.

To perform this Diviſion I divide one of the Terms of the Quantity $acd - ac^2 + abc$ which remains to be divided, by one of thoſe of the Diviſor. For Example, $acd$ by $ad$, which gives $c$ for the Quotient; I multiply this new Quotient by the whole Diviſor $ab + ad - ac$ and I ſubtract the Product $abc + acd - ac^2$ from the remaining Dividend $acd - ac^2 + abc$; and as the two Quantities are the ſame, there remains nothing to be divided; from whence I perceive that $b + c$ is the exact Quotient of $ab^2 + abd + acd - ac^2$ divided by $ab + ad - ac$ and conſequently the Value of $x$.

XLVIII.

After having performed the foregoing Diviſion, it is eaſy to perceive what Method is to be purſued in other Examples. To reduce this Operation to a regular form, the Analyſts write down the Diviſor at the right Hand of the Dividend, ſeperating them by a Line, as in Diviſion of Numbers, chuſing afterwards a Term in the Dividend diviſible by one of thoſe of the Diviſor, they write the Quotient of thoſe two Terms

General method for dividing compound quantities.

under the Diviſor, prefixing to it the Sign +, if the two Terms which are divided one by the other have the ſame Signs, but prefixing to it the Sign —, if thoſe two Terms have different Signs. This being done, they multiply the Quotient by all the Terms of the Diviſor, and write the Product under the Dividend. But as the Uſe of this Product ſhould be to ſubtract it from the Dividend, they obſerve when writing it under this Dividend to prefix to each Term a Sign contrary to that which reſults from the Multiplication.

The Product being thus wrote down, they draw a Line and perform the Reduction with the Dividend, and the Quantity which remains, is to be divided a-new by the ſame Diviſor. They chuſe as before a Term diviſible by one of thoſe of the Diviſor, and write the Term which reſults for Quotient beſide the firſt, obſerving to prefix to it the Sign +, or the Sign —, according as the two Terms that have been divided one by the other have the ſame or different Signs. They multiply afterwards this Term by all thoſe of the Diviſor, and write the Product under the Quantity to be divided, obſerving as before to change the Signs which reſult from the Multiplication. Then drawing a Line and reducing, if all the Terms do not deſtroy each other they write the Remainder under this Line, and continue the Operation after the ſame Manner until all the Terms of the Dividend have vaniſhed.

Manner of avoiding working by conjecture in diviſion.

As in this Operation, it may be ſome Times embarraſſing to chuſe among the Terms of the Dividend and the Diviſor, thoſe which ſhould ſerve to form the Terms of the Quotient. To remove all uncertainty in this Choice: Here is what the Analyſts have imagined.

What is underſtood by ordering a quantity according to a letter

They firſt chuſe at will a Letter which is found in the Dividend and Diviſor, and they diſpoſe the Terms of thoſe two Quantities in ſuch a Manner that the Terms may ſtand firſt in which this Letter has the greateſt Exponents, and thoſe next in which this Letter has the next greateſt Exponent and ſo on. Having therefore ordered the two propoſed Quantities according to the ſame Letter (it is thus this Operation is called) the Terms to be divided are no more determined by conjecture, it is always the firſt Terms of the Dividend and Diviſor which are to be choſen.

When the firſt Term of the Quotient is formed by thoſe two Terms of the Diviſor and Dividend, and the Product is wrote with different Signs under the Dividend, if this Operation introduces Terms of a different Kind from any of thoſe in the Dividend; Care is to be taken in writing the Quantity which reſults after the Reduction to place the Terms, ſo that the Quantity which remains to be divided be always ordered according to the ſame Letter as the Diviſor.

XLIX.

$$
\begin{array}{l|l}
2a^4 - 13ba^3 + 31b^2a^2 - 38b^3a + 24b^4 & 2a^2 - 3ba + 4b^2 \\
-2a^4 + 3ba^3 - 4b^2a^2 & a^2 - 5ba + 6b^2 \\
\hline
-10ba^3 + 27b^2a^2 - 38b^3a + 24b^4 & \\
+10ba^3 - 15b^2a^2 + 20b^3a & \\
\hline
+12b^2a^2 - 18b^3a + 24b^4 & \\
-12b^2a^2 + 18b^3a - 24b^4 & \\
\hline
0 &
\end{array}
$$

Application of the foregoing method to an example.

To illuſtrate this Method, we will apply it to ſome Examples. Let it be firſt propoſed to divide the Quantity $31a^2b^2 + 2a^4 + 24b^4 - 38ab^3 - 13a^3b$ by the Quantity $-3ab + 2a^2 + 4b^2$.

Having wrote down thoſe Quantities in the Manner above, where they are ordered according to the Letter $a$.

I divide the firſt Term $2a^4$ of the Dividend by the firſt Term $2a^2$ of the Diviſor, and I write the Quotient $a^2$ under the Diviſor, without prefixing any Sign to it, that is, I mak it affirmative, becauſe the Terms $2a^4$ and $2a^2$ are preceded by the ſame Signs. The Quotient $a^2$ being ſet down I multiply it by all the Terms of the Diviſor, and as this Multiplication ſhould give me for firſt Term $2a^4$ Product of $a^2$ into $2a^2$ with the Sign $+$, I ſet down this Term under the Dividend with the Sign $-$, becauſe it is to be ſubtracted.

In like Manner, becauſe the ſecond Term $3ba^3$ Product of $a^2$ into $3ba$ ſhould have the Sign $-$ by the Multiplication, I write under the Dividend $+3ba^3$ ſince it is to be ſubtracted, finally becauſe the third Term $4b^2a^2$ Product of $a^2$ into $4b^2$ ſhould have by the Multiplication the Sign $+$ I write it under the Dividend with the Sign $-$.

This being done, I draw a Line and reduce; the Quantity which remains is $-10ba^3 + 27b^2a^2 - 38b^3a + 24b^4$ which is to be divided by the ſame Dviſor $2a^2 - 3ba + 4b^2$. To perform this Diviſion, I divide the firſt Term $-10ba^3$ of the Quantity to be divided by the firſt Term $2a^2$ of the Diviſor, I write the Quotient $-5ba$ beſide $a^2$, and multiply this new Term of the Quotient by the Diviſor, and write the Terms of the Product with contrary Signs under the Quantity to be divided. After the Reduction there remains $12b^2a^2 - 18b^3a + 24b^4$ to be divided.

The firſt Term $12b^3a^2$ of this Quantity being divided by $2a^2$ of the Diviſor, gives $+6b^2$ for third Term of the Quotient, and as the Product of this third Term into the Diviſor deſtroys all thoſe of the Quantity to be divided, I conclude that the Diviſion is ended, and that $a^2 - 5ba + 6b^2$ is the Quotient ſought.

L.

$$\begin{array}{r|l}
4c^4 - 9b^2c^2 + 6b^3c - b^4 & 2c^2 - 3bc + b^2 \\
-4c^4 + 6b^2c^2 - 2b^2c^2 & 2c^2 + 3bc - b^2 \\
\hline
6bc^3 - 11b^2c^2 + 6b^3c - b^4 & \\
-6bc^3 + 9b^2c^2 - 3b^3c & \\
\hline
-2b^2c^2 + 3b^3c - b^4 & \\
+2b^2c^2 - 3b^3c + b^4 & \\
\hline
0 &
\end{array}$$

Another example.

Let it be propoſed to divide $6b^3c - b^4 - 9c^2b^2 + 4c^4$ by $-3cb + b^2 + 2c^2$, I write thoſe two Quantities in the Manner above after having ordered them according to the Letter $c$.

Dividing afterwards the two firſt Terms, I have $2c^2$ for the firſt Term of the Quotient, which being multiplyed by the Diviſor, gives, after changing the Signs, the Quantity $-4c^4 + 6bc^3 - 2b^2c^2$, which being placed under the Dividend, gives for Remainder $6bc^3 - 11b^2c^2 + 6b^3c - b^4$, in which I obſerved to place firſt the Term $6bc^3$ affected by $c^3$ introduced by the Multiplication, in order that the Quantity may remain ordered according to $c$. Dividing afterwards this firſt Term $6bc^3$ by $2c^2$, I have $3bc$ for Quotient with the Sign $+$. I multiply this new Term of the Quotient by the Diviſor, and write the Terms of the Product under the Dividend, after having changed their Signs. After the Reduction there remains $-2b^2c^2 + 3b^3c - b^4$ to be divided. The firſt Term of this Quantity being divided by that of the Diviſor, gives for third Term of the Quotient $b^2$ affected with the Sign $-$, becauſe the Terms $2b^2c^2$ and $2c^2$ have different Signs, and as the Product of this third Term into the Diviſor, deſtroys all thoſe of the Quantity to be divided, I conclude that the Diviſion is ended, and that $2c^2 + 3bc - b^2$ is the Quotient ſought.

LI.

If in ordering the Dividend and the Diviſor according to the ſame Letter, there occur ſeveral Terms in which this Letter has the ſame Exponent, we would fall into the ſame Inconveniency which we propoſed to avoid; unleſs thoſe Terms be ordered again according to another Letter, common to the two Quantities.

Let us ſuppoſe, for Example, that the Dividend being ordered according to the Letter $d$, the firſt Terms are, $3ac^2d^3 - c^3d^3 - 3a^2cd^3 + a^3d^3$, and thoſe of the Diviſor, ordered according to the ſame Letter $a^2d^2 + c^2d^2 - 2acd^2$. Arranging theſe two Quantities thus $a^3d^3 - 3ca^2d^3 + 3c^2ad^3 - c^3d^3$, $a^2d^2 - 2cad^2 + c^2d^2$, that is, ordering them according to the Letter $a$, the Diviſion will be performed without being liable to be embarraſſed in chuſing in the Dividend and Diviſor the Terms which ſhould ſerve to form thoſe of the Quotient.

To accuſtom Beginners to be attentive to thoſe Particularities in Diviſion, here follow more Examples.

| | *Dividend.* | *Diviſor.* |
|---|---|---|
| | $y^4 \bullet -3\frac{1}{2}a^2y^2 + 3a^3y - \frac{1}{2}a^4$ | $y^2 - 2ay + a^2$ |
| Product. | $-y^4 + 2ay^3 - a^2y^2$ | $y^2 + 2ay - \frac{1}{2}a^2$ |
| Reduction. 1. | $+2ay^3 - 4\frac{1}{2}a^2y^2 + 3a^3y - \frac{1}{2}a^4$ | |
| Product. | $-2ay^3 + 4a^2y^2 - 2a^3y$ | |
| Reduction. 2. | $-\frac{1}{2}a^2y^2 + a^3y - \frac{1}{2}a^4$ | |
| Product. | $+\frac{1}{2}a^2y^2 - a^3y + \frac{1}{2}a^4$ | |
| Reduction. 3. | $0$ | |

| | *Dividend.* | *Diviſor.* |
|---|---|---|
| | $y^6 + aay^4 + b^4yy - a^6$<br>$-2bby^4 - a^4yy - 2a^4bb$<br>$-aab^4$ | $yy - aa - bb$<br>$y^4 + 2aayy + a^4$<br>$-bbyy + aabb$ |
| Product. | $-y^6 + aay^4$<br>$-bby^4$ | |
| Reduction. 1. | $+2aay^4 + b^4yy - a^6$<br>$-bby^4 - a^4yy - 2a^4bb$<br>$-aab^4$ | |
| Product. | $-2aay^4 + 2a^4yy$<br>$+2a^2b^2yy$ | |
| Reduction. 2. | $-bby^4 + b^4yy - a^6$<br>$+a^4yy - 2a^4bb$<br>$+2aabbyy - aab^4$ | |
| Product. | $+bby^4 - aabbyy$<br>$-b^4yy$ | |
| Reduction. 3. | $+a^4yy - a^6$<br>$+aabbyy - 2a^4bb$<br>$-aab^4$ | |
| Product. | $-a^4yy + a^6$<br>$+a^4bb$ | |
| Reduction. 4. | $+aabbyy - a^4bb$<br>$-aab^4$ | |
| Product. | $-aabbyy + a^4bb$<br>$+aab^4$ | |
| Reduction. 5. | $0$ | |

LII.

In the Solution of the foregoing Problems, we had occaſion to employ only one unknown Quantity, becauſe in thoſe Problems there was, properly ſpeaking, only one Quantity to be inveſtigated. But as we advance in the Science of ſpecious Arithmetick, there occur Problems, to ſolve which, it is requiſite to employ ſeveral unknown Quantities, we ſhall therefore proceed to explain how they are to be managed.

Problem in which two unknown quantities are employed.

*There being given the ſpecifick Gravities of two Subſtances which are mixed together, the Bulk and Weight of the Mixture, to find the Quantity of each of thoſe Subſtances which compoſe the Mixture.*

Let the Number of cubick Inches contained in this Mixture, or in general, its Bulk after whatever Manner it is meaſured, be expreſſed by $a$, and its Weight by $b$.

Let the Quantity of the firſt Subſtance contained in the Mixture, for Example, the Number of cubick Inches of this Subſtance be expreſſed by $x$.

Let the Weight of a cubick Inch of this Subſtance, or in general, its ſpecifick Gravity be expreſſed by $c$.

Let the Quantity of the ſecond Subſtance be expreſſed by $y$, and its ſpecifick Gravity by $d$.

The Weight of the Quantity of the firſt Subſtance contained in the Mixture will be expreſſed by $cx$.

For if $x$ expreſſes the Number of cubick Inches of this Subſtance, and $c$ the Weight of each cubick Inch, their whole Weight will be the Product of thoſe two Numbers. In like Manner the Weight of the Quantity of the ſecond Subſtance will be expreſſed by $dy$.

But thoſe two Weights added together is equal to the Weight of the Mixture, hence we have the Equation $cx + dy = b$.

But this Equation is not ſufficient to ſolve the Problem; for diſengaging one of the unknown Quantities, for Example, $x$, we find $x = \frac{b - dy}{c}$, whereby $x$ cannot be determined unleſs we ſuppoſe $y$ to be known. There is therefore ſome other Operation to be performed to diſcover $y$. To arrive at which we muſt examine whether all the Particulars expreſſed in the Queſtion have been attended to, or to ſpeak in the Language of the Analyſts, if all the Conditions of the Problem have been fulfilled; upon Examination it will appear that only one of the Conditions has been expreſſed, and the Condition implying that the Quantity of the firſt Subſtance, added to the Quantity of the ſecond, ſhould be equal to the Bulk of the Mixture has not been employed. Hence this ſecond Condition furniſhes the Equation $x + y = a$, which, as the firſt, determines the Value of $x$ only by Means of that of $y$, by giving $x = a - y$.

But if we cannot by either of thoſe two Equations, taken ſeperately, find $x$ independent of $y$, by employing them both together we are enabled to determine $y$. For ſince each of thoſe two Equations gives a Value of $x$, we may equal theſe two Values, which gives the Equation $\frac{b-dy}{c} = a-y$, from which is deduced by the foregoing Methods $b-dy = ac-cy$, or $ac-b = cy-dy$, or finally $y = \frac{ac-b}{c-d}$.

$y$ being known it eaſily appears that $x$ which is expreſſed by $a-y$, or $\frac{b-dy}{c}$ is alſo known. We have therefore only to ſubſtitute in which of thoſe two Quantities we pleaſe, in the firſt $a-y$, for Example, inſtead of $y$, $\frac{ac-b}{c-d}$, and we will have $a - \frac{ac-b}{c-d}$ for the Value of $x$.

On examining the foregoing Value $a - \frac{ac-b}{c-d}$ it will eaſily be perceived that it can be reduced; for if we put $a$ to the ſame Denominator as the Fraction $\frac{ac-b}{c-d}$, we muſt multiply it by $c-d$, which gives $\frac{ac-ad}{c-d}$ inſtead of $a$; we have no more to do than to ſubtract from this Fraction the ſecond $\frac{ac-b}{c-d}$. To this End ſubtracting their Numerators, and dividing the Remainder by the common Denominator, we will have $\frac{ac-ad-ac+b}{c-d}$ or $\frac{b-ad}{c-d}$ for the reduced Value of $x$.

The Quantities required therefore of the firſt and ſecond Subſtance which compoſe the Mixture are expreſſed, one by $\frac{ac-b}{c-d}$ and the other by $\frac{b-ad}{c-d}$, conſequently the Problem is ſolved.

LIII.

If inſtead of ſubſtituting the Value $\frac{ac-b}{c-d}$ of $y$ in $a-y$, we ſubſtituted it in $\frac{b-dy}{c}$, which alſo expreſſes $x$. There would reſult $\frac{b-d\times\left(\frac{ac-b}{c-d}\right)}{c}$ which at firſt View ſeems to be a different Value from $\frac{b-ad}{c-d}$. But as we know that the two Values $a-y$,

and $\frac{b - dy}{c}$ of $x$ are equal, thoſe two Values of $x$ expreſſed in known Quantities muſt coincide. The Manner of reducing one to the other is as follows.

I firſt give the Denominator $c - d$ to the Letter $b$, by multiplying it by $c - d$, that is, by ſubſtituting $\frac{bc - bd}{c - d}$ for $b$, and then the foregoing Quantity $\frac{b - \frac{d \times (ac - b)}{c - d}}{c}$ will be transformed into $\frac{\frac{bc - bd - d \times (ac - b)}{c - d}}{c}$, or $\frac{bc - bd - d \times (ac - b)}{cc - dc}$; but inſtead of $d \times (ac - b)$ we may write $acd - bd$, and as this Quantity ſhould be ſubtracted from $bc - bd$, the foregoing Quantity $\frac{bc - bd - d \times (ac - b)}{cc - dc}$ when reduced is $\frac{bc - dca}{cc - dc}$, which by dividing the Numerator and Denominator by the ſame Quantity $c$, at length becomes $\frac{b - da}{c - d}$ the ſame Value as before.

LIV.

Application of the foregoing ſolution to an example.

To apply this general Solution to an Example; let the Mixture be ſuppoſed to be compoſed of Gold and Silver, its Weight to be 30 Ounces, its Bulk 3 cubick Inches, the Weight of a cubick Inch of Gold $12\frac{2}{3}$ Ounces, and that of a cubick Inch of Silver $6\frac{2}{5}$ Ounces.

We will have $a = 3$, $b = 30$, $c = 12\frac{2}{3}$, $d = 6\frac{2}{5}$. Subſtituting thoſe Values in the two general Formulas $x = \frac{b - da}{c - d}$ and $y = \frac{ac - b}{c - d}$, they will become $x = \frac{21}{13}$ and $y = \frac{18}{13}$, that is, the Mixture will contain $\frac{21}{13}$ cubick Inches of Gold, and $\frac{18}{13}$ cubick Inches of Silver.

LV.

It is eaſy to perceive after what we have ſeen in the foregoing Problem, that as often as two unknown Quantities are employed in a Queſtion, there muſt be two Equations to diſengage them, and when there are two Quantities required in a Problem, there muſt be two Conditions given to determine them, in order to deduce from thoſe two Conditions two Equations. To explain the Method of employing thoſe Conditions we have added the following Problem.

LVI.

Problem in which two unknown

*Two Pipes, each of which run uniformly together, filled a Ciſtern* a, *the one running during the Time* b, *the other during the Time* c; *the two ſame*

*Pipes filled another Ciſtern* d, *the firſt running during the Time* e, *the ſecond during the Time* f. *The Diſcharge of each Pipe is required.*

quantities are employed.

Let $x$ and $y$ expreſs thoſe Diſcharges, that is, for Example, the Number of Hogſheads that each of thoſe two Pipes furniſh in a Day, ſuppoſing the Ciſterns $a$ and $d$ to be meaſured in Hogſheads, whilſt the Times $b$, $c$, $e$, $f$ are counted in Days.

The Quantity of Water furniſhed by the firſt Pipe, during the Time $b$, will be expreſſed by $bx$; and the Quantity of Water furniſhed by the ſecond Pipe in the Time $c$, will be expreſſed by $cy$. But thoſe two Quantities of Water by the firſt Condition of the Problem ſhould be equal to the Ciſtern $a$, hence we have $bx + cy = a$.

In like Manner the Quantities of Water furniſhed by the ſame Pipes during the Times $e$, $f$ will be expreſſed by $ex$, $fy$, and conſequently the ſecond Condition will give $ex + fy = d$.

We have no more to do now than to deduce from thoſe two Equations the Values of $x$ and of $y$, which will be performed, as in the foregoing Problem, by deducing a Value of $x$ from each of thoſe two Equations, and then putting theſe two Values equal to each other.

The firſt will be $\frac{a - cy}{b}$, the ſecond $\frac{d - fy}{e}$. Putting theſe two Values equal to each other, there will ariſe $\frac{a - cy}{b} = \frac{d - fy}{e}$ or $ae - cey = bd - bfy$, or $bfy - cey = bd - ae$, or finally $y = \frac{bd - ae}{bf - ce}$. Subſtituting this Value of $y$ in one of the two Values of $x$, in $\frac{a - cy}{b}$, for Example, there will reſult $x = \frac{a - c \times (\frac{bd - ae}{bf - ce})}{b}$ or $x = \frac{a \times (bf - ce) - c \times (bd - ae)}{b \times (bf - ce)}$ by giving the firſt Term $a$ the ſame Denominator as the ſecond, and by multiplying the two Denominators, one by the other.

Performing afterwards the Multiplications indicated in this Value, and reducing, there will ariſe $x = \frac{af - cd}{bf - ce}$.

Wherefore when the particular Values of $a$, $b$, $c$, $d$, $e$, $f$, are aſcertained, we have only to ſubſtitute them in thoſe two general Values of $x$ and of $y$, to obtain any particular Solution we pleaſe.

Inſtead of diſengaging $x$ in the two foregoing Equations, and putting the two Values which ariſe equal to each other, in order to obtain $y$, it is manifeſt that if we diſengaged $y$, and afterwards put the two Values which ariſe equal to each other, to obtain $x$, the Reſult would be the ſame.

LVII.

Example of the foregoing problem in numbers.

To make some Application of this Problem, let us suppose that the first Pipe running five Days, and the second four, filled a Cistern which contains 330 Hogsheads. That afterwards the first Pipe running two Days, and the second three, filled a Reservoir containing 195 Hogsheads.

In this Case $a = 330$, $b = 5$, $c = 4$, $d = 195$, $e = 2$, $f = 3$, and consequently $af - dc = 210$, $bf - ce = 7$, $bd - ae = 315$, consequently $x = \frac{af - dc}{bf - ce} = \frac{210}{7} = 30$, and $y = \frac{bd - ae}{bf - ce} = \frac{315}{7} = 45$. Hence the first Pipe in this Example furnishes 30 Hogsheads a Day, and the second 45.

LVIII.

Another example.

Let us now suppose that the first Pipe running during 3 Days, and the second during seven, filled a Cistern containing 190 Hogsheads, that afterwards the first running 4 Days, and the second 6, they filled a Cistern containing 120 Hogsheads.

In this Case $a = 190$, $b = 3$, $c = 7$, $d = 120$, $e = 4$, $f = 6$, and consequently $af - dc = 300$, $bf - ce = -10$, $bd - ae = -400$, which will give $x = \frac{af - dc}{bf - ce} = \frac{300}{-10}$, and $y = \frac{db - ae}{bf - ce} = \frac{-400}{-10}$.

Singularity of the expressions arrived at in this example.

The first Time the Analysts found similar Values, that is, negative Quantities divided by negative Quantities, and positive Quantities divided by negative ones; without Doubt they were embarrassed to know what they meant, and probably resumed the Question, in order to avoid those Kinds of Division, and in the present Case would have proceeded thus.

Method of discovering their meaning.

They would have resumed the two general Equations $bx + cy = a$, and $ex + fy = d$, and substituting in those Equations for $a, b, c, d, e, f$, the Values which those Letters have in this Example, there would result $3x + 7y = 190$ and $4x + 6y = 120$, deducing from those two Equations $x = \frac{190}{3} - \frac{7y}{3}$, and $x = \frac{120}{4} - \frac{3}{2}y$, they would have put them equal to each other, which would have given $\frac{190}{3} - \frac{7}{3}y = 30 - \frac{3}{2}y$, or $\frac{7}{3}y - \frac{3}{2}y = \frac{190}{3} - 30$, or $y = 40$.

Substituting afterwards this Value of $y$ in $30 - \frac{3}{2}y$ Value of $x$, they would have found $x = 30 - 60$, that is, $x = -30$. In this Manner they would have assured themselves that the Quotient arising from $-400$ divided by $-10$ is $+40$, and that arising from 300 divided by $-10$ is $-30$.

LIX.

General Theorems concerning the signs of quotients or products.

They ſoon after eſtabliſhed as general Principles that + divided by +, produces +, that + divided by — produces —, that — divided by + produces — that — divided by — produces +, and the ſame for Multiplication.

Thoſe Principles were the more eaſily diſcovered as they reſulted from the Obſervations which muſt neceſſarily have been made on the Signs found for the Terms of the Products and Quotients, in applying the Rules given for the Multiplication and Diviſion of compound Quantities, but it is probable that the firſt Analyſts did not adopt them until after they had verified them in ſeveral Examples.

LX.

Demonſtration that $-b$ into $-d$ is $+bd$ when thoſe quantities are not preceded by others.

To aſſure ourſelves that the Multiplication of — by — ſhould produce + in the Product, let us ſee what Aſſiſtance we can draw from the general Method of Multiplications delivered Art. XLV. According to this Method, we ſee plainly that the Product of a Quantity, ſuch as $a-b$ into another $c-d$ ſhould be $ac-bc-ad+bd$, and conſequently we ſee at the ſame Time that the Term $bd$ which ariſes from the Multiplication of $b$ into $d$ has the Sign +, whilſt its Factors $b$ and $d$ have the Sign —. It remains therefore to know if when two negative Quantities, ſuch as $-b$ and $-d$ are not preceded by any poſitive Quantities, their Product will be ſtill $+bd$, the Truth of which will eaſily appear by obſerving that the Method by which we found the Product of $a-b$ into $c-d$ to be $ac-bc-ad+bd$ was not reſtrained to any particular Value of $a$ or $c$, and conſequently took Place when thoſe Quantities were equal to nothing. But in this Caſe, the Product $ac-bc-ad+bd$ is reduced to $+bd$, wherefore $-b\times-d=+bd$.

LXI.

The other caſes are demonſtrated in like manner.

As to the other Caſes, that is, with Reſpect to the Multiplication and Diviſion of + by —, they will be proved after the ſame Manner.

LXII.

How the negative value found, ſolves the problem

To return now to our laſt Application of the foregoing Problem, it is to be obſerved that after having found $x=-30$ and $y=+40$ we ſhould be ſtill at a Loſs to know what this Value of $x$ meant. The Method the firſt Analyſts probably would have taken to ſatisfy themſelves in this Point, would be to return back to the Conditions of the Problem or which comes to the ſame, they would reſume the two Equations $3x+7y=190$, and $4x+6y=120$, and try how the Values $-30$ and $+40$ of $x$ and of $y$, anſwered thoſe Equations, we find firſt, that in this Caſe $3x=-90$ and $7y=280$, conſequently $3x+7y=-90+280$, which in effect is equal to 190. In like Manner $4x+6y$ is found to be $-120+240$ which is reduced to 120.

Having therefore discovered how the Values $-30$ and $+40$ of $x$ and of $y$ answer the Equations $3x + 7y = 190$ and $4x + 6y = 120$, we perceive at the same Time how they answer the Conditions of the Problem; for since the Use that have been made of the Quantities $3x$ and $4x$, which express the Quantities of Water discharged by the first Pipe in the first and second Operation, was to subtract them from $7y$ and from $6y$, which express the Quantities of Water furnished in the same Operations by the second Pipe. The first Pipe must be considered in this Case as depriving the Cisterns of Water instead of furnishing any, as it did in the other Example, and as it was supposed in expressing the Conditions of the Problem.

We have here an Example of the Generality of the analytick Art, whereby we find in a Question Cases which we did not foresee could be included in it.

LXIII.

Unknown quantities becoming negative are of an opposite kind from what they were supposed in the expression of the problem, as also known quantities.

In almost every Question solved after a general Manner, the Analysts found Cases similar to the foregoing, and they always concluded, that when the Value of the unknown Quantity became negative, the Quantity expressed by it should be considered as being of an opposite Kind from what it was supposed in expressing the Conditions of the Problem.

What has been said with Respect to unknown Quantities, is equally applicable to known Quantities, that is, when a general Solution is applied to any particular Case, if any of the given Quantities $a$, $b$, &c. in the Problem are negative.

LXIV.

Example of the use of known quantities made negative.

Let it be proposed, for Example, to find what should be in the foregoing Problem the Discharges of two Pipes that the first furnishing Water during 3 Days, and the second during four Days, may fill a Cistern containing 320 Hogsheads, and that the second Pipe afterwards furnishing Water during six Days whilst the first discharges it during 3 Days, may fill a Cistern containing 180 Hogsheads.

We have only to put in the general Solution $a = 320$, $b = 3$, $c = 4$, $d = 180$, $e = -3$, $f = 6$.

And there will result $dc = 720$, $af = 1920$, $ce = 18$, $bf = -12$, $ae = -960$, $db = 540$, and consequently $af - dc = 1200$, $bf - ce = 30$, $db - ae = 1500$, which gives $x = \frac{af - dc}{bf - ce} = 40$, and $y = \frac{db - ae}{bf - ce} = 50$.

From whence it appears that the Discharge of the first Pipe is 40 Hogsheads a Day either to carry away the Water as in the second Operation, or to furnish it as in the first, and the Discharge of the second 50 Hogsheads a Day, which it furnishes in both Operations.

LXV.

To accuſtom Beginners to the Manner of extending the Solutions of Problems to thoſe Caſes in which the given Quantities are of an oppoſite Kind from what they have been ſuppoſed to be at firſt, I ſhall give another Example taken from the Problem of Art. XXIV. in which it is required to know where two Meſſengers will meet, and will endeavour to deduce from the general Solution, the Solution of the following Caſe.

Another example in which the known quantities are made negative.

Two Meſſengers at 50 Miles Diſtance from each other, one being, for Example, at *Newry*, the other at *Dublin*. The firſt ſets out from *Newry* for *Dublin* at 8 in the Evening, running at the Rate of 4 Miles an Hour, the ſecond ſets out the ſame Day from *Dublin* for *Newry*, at 11 in the Morning, running 3 Miles an Hour, it is required to know at what Diſtance from *Dublin* they will meet.

Comparing this Caſe with the general Expreſſion of the Problem, it appears firſt, that the Letter $c$ which denoted the Number of Miles run by the firſt Meſſenger in a given Time, ſhould be negative, ſince in the general Solution the firſt Meſſenger was ſuppoſed to recede from, and, in this Caſe, he goes to meet the ſecond. It alſo appears that the Letter $b$, which expreſſes the Number of Hours that the firſt Meſſenger ſet out ſooner than the ſecond, ſhould be alſo negative, becauſe in this Caſe he ſet out later than the ſecond.

Hence we have only to put in the general Formula - - - - $x = \frac{a\,d\,e + b\,c\,e}{d\,e - c\,f}$, $a = 50$, $b = -9$, $c = -4$, $d = 1$, $e = 3$, $f = 1$, and we will have $x = \frac{50 \times 1 \times 3 - 9 \times -4 \times 3}{1 \times 3 + 4 \times 1}$, $= \frac{150 + 108}{3 + 4} = \frac{258}{7} = 36\frac{6}{7}$, by which we learn that when the Meſſenger from *Dublin* has run $36\frac{6}{7}$ Miles, he will meet the Meſſenger from *Newry*.

LXVI.

Two equations whatſoever of the firſt degree, including two unknown quantities may be reduced to the foregoing ones.

One of the principal Advantages of ſpecious Arithmetick, and which ſhews the Utility of aſſuming at Will the Signs of the given Quantities in the general Solution of Problems, is the reducing to the Solution of Equations expreſſed indefinitely all thoſe in which the unknown Quantities are diſpoſed after the ſame Manner, for Example, by Means of the two Equations $b\,x + c\,y = a$, and $e\,x + f\,y = d$ ſolved Article LVI. any two Equations whatſoever of the firſt Degree may be ſolved, provided they include only two unknown Quantities.

Example.

Let, for Example, the two Equations $m\,n\,x = p^2\,y - b\,b\,g$, and $m\,n\,y = p^3 - n\,n\,x$ be propoſed to be ſolved. To compare them with the two firſt I write them thus $m\,n\,x - p^2\,y = -b\,b\,g$, and

$n\,n\,x + n\,m\,y = p^3$, then comparing them with the two Equations $b\,x + c\,y = a$ and $e\,x + f\,y = d$. The first with the first, and the second with the second, we will have $b = m\,n$, $c = -p^2$, $a = -b\,b\,g$, $e = n^2$, $f = m\,n$, $d = p^3$.

Which will give $c\,d = -p^5$, $a\,f = -m\,n\,b^2\,g$, $c\,e = -p^2\,n^2$, $b\,f = m^2\,n^2$, $a\,e = -b^2\,n^2\,g$, $b\,d = m\,n\,p^3$, and consequently $a\,f - c\,d = -m\,n\,b^2\,g + p^5$, $b\,f - c\,e = m^2\,n^2 + p^2\,n^2$, $b\,d - a\,e = m\,n\,p^3 + b^2\,n^2\,g$.

Now substituting those Values in the general Formulas $x = \frac{a\,f - c\,d}{b\,f - c\,e}$, and $y = \frac{b\,d - a\,e}{b\,f - c\,e}$, we will have at length $x = \frac{p^5 - m\,n\,b^2\,g}{m^2\,n^2 + p^2\,n^2}$, $y = \frac{m\,n\,p^3 + b^2\,n^2\,g}{m^2\,n^2 + p^2\,n^2}$.

LXVII.

Another example. Let the Equations $\frac{3\,m\,p\,x}{p-q} = \frac{p^2\,y}{p+q} - \frac{2\,n\,q}{p-q}$ and $m\,x + (p+q) \times y = \frac{n\,q}{p-q}$ be proposed to be solved, giving the first this Form $\frac{3\,m\,p\,x}{p-q} - \frac{p^2}{p+q}\,y = -\frac{2\,n\,q}{p-q}$, and comparing it with the general Equation $b\,x + c\,y = a$, $b = \frac{3\,m\,p}{p-q}$, $c = -\frac{p^2}{p+q}$, $a = -\frac{2\,n\,q}{p-q}$, comparing the second with the Equation $e\,x + f\,y = d$, we will have $e = m$, $f = p + q$, $d = \frac{n\,q}{p-q}$, and those Values being substituted in the Formula $x = \frac{a\,f - c\,d}{b\,f - c\,e}$, gives $x = \frac{-\frac{2\,n\,q}{p-q} \times (p+q) + \frac{p^2}{p+q} \times \frac{q\,n}{p-q}}{\frac{3\,m\,p}{p-q} \times (p+q) + \frac{p^2}{p+q} \times m}$.

To reduce this Quantity, I first multiply the Numerator and Denominator of the Fraction $\frac{-2\,n\,q \times (p+q)}{p-q}$ by $p + q$, which changes it into $\frac{-2\,n\,q \times (p+q)^2}{(p-q) \times (p+q)}$, by this Means the whole Numerator of the Value of $x$ becomes $\frac{-2\,n\,q \times (p+q)^2 + p^2 \times q\,n}{(p+q) \times (p-q)}$ or $\frac{[p^2 - 2 \times (p+q)^2] \times n\,q}{(p-q) \times (p+q)}$ or $-q\,n \times \frac{+p^2 + 4\,p\,q + 2\,q^2}{(p-q) \times (p+q)}$.

I operate afterwards on the Denominator of the Value of $x$, reducing its two Parts to the same Denominator.

Which gives $\frac{3mp \times (p+q)^2 + p^2 \times (p-q) \times m}{(p+q) \times (p-q)}$, or $\frac{mp \times [p^2 - pq + 3 \times (p+q)^2]}{(p+q) \times (p-q)}$, or $\frac{mp \times (4p^2 + 5pq + 3q^2)}{(p+q) \times (p-q)}$.

Thoſe two Operations change the foregoing Value of $x$ into $\frac{-qn \times \frac{p^2 + 4pq + 2q^2}{(p+q) \times (p-q)}}{mp \times \frac{4p^2 + 5pq + 3q^2}{(p+q) \times (p-q)}}$; but as the Numerator and Denominator of this Fraction are each divided by $(p+q) \times (p-q)$, I take away this Diviſor, and the Value of $x$ becomes $\frac{-qn \times (p^2 + 4pq + 2qq)}{mp \times (4p^2 + 5pq + 3q^2)}$, or $-\frac{qn}{mp} \times \frac{pp + 4pq + 2qq}{4pp + 5pq + 3qq}$, or finally $x = \frac{-nqpp - 4pqqn - 2nq^3}{4mp^3 + 5mp^2q + 3mpqq}$. Subſtituting in like Manner the ſame Values of $a$, $b$, $c$, &c. in the general Formula $y = \frac{bd - ae}{bf - ce}$, we will have $y = \frac{\frac{3mp}{p-q} \times \frac{nq}{p-q} + \frac{2nq}{p-q} \times m}{\frac{3mp}{p-q} \times (p+q) + \frac{pp}{p+q} \times m}$ whoſe Numerator I reduce to this Form $\frac{3mnpq + 2mnq \times p - q}{(p-q)^2}$ by multiplying the Numerator & Denominator of the Fraction $\frac{2mnq}{p-q}$ by $p - q$. I afterwards reduce this new Form and it becomes $\frac{mnq \times (5p - 2q)}{(p-q)^2}$.

As to the Denominator of the Value of $y$, it being the ſame as that of $x$, it will be reduced after the ſame Manner, and conſequently we will have $y = \frac{mnq \times \frac{5p - 2q}{(p-q)^2}}{mp \times \frac{4pp + 5pq + 3qq}{(p+q) \times (p-q)}}$, which by taking away the common Diviſors $p - q$, and reducing, there reſults $y = \frac{qn \times \frac{5p - 2q}{p-q}}{p \times \frac{4pp + 5pq + 3qq}{p+q}}$, which, by putting the Diviſor $p + q$ at Top, and the Diviſor $p - q$ at Bottom, according to the Rules of the Diviſion of Fractions, will at length become

$$y = \frac{qn \times (p+q) \times (5p - 2q)}{p \times (p-q) \times (4p^2 + 5pq + 3q^2)}.$$

or $y = \frac{-2q^3n + 5p^2qn + 3pq^2n}{-2p^2q^2 + 4p^4 + p^3q - 3pq^3}$.

LXVIII.

Another manner of solving the foregoing example.

If we had freed the Equations propoſed in this Example from Fractions, before we compared them with the general Formulas, they would have been more readily ſolved.

Let the two Members of the Equation $\frac{3mpx}{p-q} = \frac{ppy}{p+q} - \frac{2qn}{p-q}$ or $\frac{3mpx}{p-q} - \frac{ppy}{p+q} = -\frac{2nq}{p-q}$ be multiplied by $pp - qq$, Product of the two Diviſors $p - q$, $p + q$, and we will have the Equation $(3mpp + 3mpq) \times x + (ppq - p^3) \times y = -2npq - 2nqq$.

In like Manner, let the two Members of the Equation $mx + (p+q) \times y = \frac{nq}{p-q}$ be multiplied by $p - q$, and we will have $mp - mq \times x + (pp - qq) \times y = qn$.

Now comparing thoſe two new Equations with the two general Formulas $b = 3mpp + 3mpq$, $c = ppq - p^3$, $a = -2npq - 2nq^2$, $e = mp - mq$, $f = p^2 - q^2$, $d = qn$.

From whence is deduced $cd = p^2q^2n - p^3qn$, $af = -2np^3q - 2np^2q^2 + 2pq^3n + 2nq^4$, $ae = 2nmq^3 - 2nmp^2q$, $bd = 3mnp^2q + 3mnpq^2$, $ce = 2p^3qm - mp^4 - mp^2q^2$, $bf = 3mp^3q + 3mp^4 - 3mpq^3 - 3mp^2q^2$, hence

$af - cd = 2nq^4 + 2npq^3 - 3np^2q^2 - np^3q$,

$bf - ce = -3mpq^3 - 2mp^2q^2 + mp^3q + 4mp^4$,

$bd - ae = -2nmq^3 + 3mnpq^2 + 5mnp^2q$,

which gives $x = \frac{-np^3q - 3np^2q^2 + 2pq^3n + 2nq^4}{mp^3q + 4mp^4 - 3mpq^3 - 2mp^2q^2}$,

or $x = \frac{qnp^3 + 3pnqqn - 2npq^3 - 2nq^4}{2mpqqq - 4mp^4 - mp^3q + 3mpq^3}$,

and $y = \frac{2nmq^3 - 5mnp^2q - 3mnpqq}{2mpqqq - 4mp^4 - mp^3q + 3mpq^3}$.

LXIX.

Compariſon of the two foregoing ſolutions.

Comparing actually thoſe two Values of $x$ and of $y$ with thoſe found in the foregoing Article, firſt the Identity of the two Values of $y$ is eaſily perceived. As to the Values of $x$ to diſcover how the former can be the ſame with the latter, we are to obſerve, that the Equality which ſhould ſubſiſt between thoſe two Expreſſions, neceſſarily ſuppoſes that the Numerator $qnp^3 + 3np^2q^2 - 2npq^3 - 2nq^4$ of the latter contains the Numerator $-qnp^2 - 4pq^2n - 2q^3n$ of the former, after the ſame Manner that the Denominator $2mp^2q^2 - 4mp^4 - mp^3q + 3mpq^3$ of the latter contains the Denomina-

tor $4p^3mq + 5p^2qm + 3mpq^2$ of the former, and dividing the ſecond Numerator by the firſt, we find in effect the ſame Quotient $p - q$, as reſults by dividing the ſecond Denominator by the firſt, that is, the Expreſſion $\frac{qnp^3 + 3p^2q^2n - 2npq^3 - 2nq^4}{2mp^2q^2 - 4mp^4 - mp^3q + 3mpq^3}$ is changed into $\frac{p-q \times (-nqp^2 - 4pq^2n - 2nq^3)}{p-q \times (4mp^3q + 5mp^2q + 3mpq^2)}$, or by taking away $p - q$, becomes $\frac{-nqp^2 - 4pq^2n - 2nq^3}{4mp^3q + 5mp^2q + 3mpq^3}$.

LXX.

It was eaſy to diſcover the Method employed to reduce the more complex Value of $x$ to the moſt ſimple, when both one and the other of thoſe two Expreſſions were known; but if the more complex Value was only known, and it was propoſed to reduce it, to its loweſt Terms, it would have been much more embarraſſing, ſince we would not know by what Quantity to divide the Numerator and Denominator of the Fraction. Now as it would be an Imperfection in the Solution of a Problem that a Quantity ſhould be reducible and not reduced, the Analyſts have ſought a Method for reducing Fractions to their leaſt Terms, or which comes to the ſame, a Method for finding the greateſt common Diviſor of any two given Quantities.

To explain what they have imagined in this Reſpect, let us firſt ſuppoſe thoſe two Quantities to be Numbers; let it be propoſed, for Example, to find the greateſt common Diviſor of the Numbers 637 and 143, or to reduce the Fraction $\frac{637}{143}$, to its leaſt Terms.

Dividing 637 by 143, the Quotient is 4 and there remains 65, that is, the Fraction $\frac{637}{143}$ is changed into $4 + \frac{65}{143}$, hence the next Step is to reduce the Fraction $\frac{65}{143}$, or to find the greateſt common Diviſor of the Numbers 143 and 65. For when this Number is found, it is manifeſt that it will be the greateſt common Diviſor of the Numbers 637 and 143, ſince we can not reduce the Fraction $\frac{65}{143}$ to its loweſt Terms, without reducing at the ſame Time $4 + \frac{65}{143}$ or $\frac{637}{143}$ to its leaſt Terms alſo.

The two Numbers 143 and 65, on which we are now to operate, being much ſimpler than the two firſt 637, 143, it is eaſy to perceive that the Difficulty is leſſened, and that proceeding after the ſame Manner it will be ſtill leſſened. Inſtead of the Fraction $\frac{65}{143}$ to be reduced, I write $\frac{143}{65}$, not that thoſe Fractions are the ſame; but becauſe one cannot be reduced without the other being reduced alſo after the ſame Manner. To reduce $\frac{143}{65}$, I divide 143 by 65, and the Quotient is 2, the Remainder being 13. We have therefore no more to do according to the ſame Principle, than to find the greateſt common Diviſor of 13 and 65, becauſe the greateſt common Diviſor of thoſe two Numbers,

will be also that of 143 and 65, since the Fraction $\frac{143}{65}$ is changed into $2 + \frac{13}{65}$.

Now the greatest common Divisor of 13 and 65 is 13 itself, since it divides 65 exactly. Wherefore 13 is also the greatest common Divisor of 143 and 65, and consequently of the proposed Numbers 637, 143. In effect 637 is $49 \times 13$ and 143, $11 \times 13$, from whence is deduced $\frac{637}{143} = \frac{49}{11}$ an irreducible Fraction.

LXXI.

General method for finding the greatest common divisor of two numbers.

It is easy to perceive that the Method employed in the foregoing Example, is applicable to any Numbers whatsoever. Let $A$ and $B$ express in general any two Numbers, and let $a$ be the Quotient arising from the Division of the first by the second, and $C$ the Remainder, the Question is reduced to find the greatest common Divisor of $B$ and $C$; then $b$ being supposed to be the Quotient resulting from the Division of $B$ by $C$, and $D$ the Remainder, we have no more to do than to find the greatest common Divisor of $C$ and $D$, that is, to divide $C$ by $D$, and if there is a Remainder to divide $D$ by it, and to proceed thus continually dividing until we find two Numbers, the least of which is contained exactly in the greatest, and this Number which is contained exactly in the other, will be the greatest common Divisor of the two first Numbers $A$ and $B$.

This Rule in its full Extent, is founded, as, in the foregoing Example, on this Principle, that the Fraction $\frac{A}{B}$ being changed into $a + \frac{C}{B}$, cannot be reduced until $\frac{C}{B}$ is reduced, and that $\frac{C}{B}$ cannot be reduced but after the same Manner that $\frac{B}{C}$, and that $\frac{B}{C}$ being changed into $b + \frac{D}{C}$ cannot be reduced until $\frac{D}{C}$ is reduced, and so on.

LXXII.

We now proceed to shew what Alterations are to be made in this Method to render it applicable to specious or algebraic Quantities, to understand which, let us first take an Example.

Let it be proposed to find the greatest common Divisor of the Quantities $3a^3 - 3baa + bba - b^3$, and $4aa - 5ba + bb$. We are, according to the foregoing Method to divide the first of those two Quantities by the second; but as the Division cannot be performed, the first Term $3a^3$ of the Dividend not containing exactly the first Term of the Divisor, I multiply the first Quantity by 4, observing that as 4 is not a Divisor of the second Quantity $4aa - 5ba + bb$, the

greateſt common Diviſor of $12a^3 - 12baa + 4bba - 4b^3$, and $4aa - 5ba + bb$ muſt be alſo the greateſt common Diviſor of $3a^3 - 3baa + bba - b^3$ and $4aa - 5ba + bb$.

I divide then, according to the foregoing Rules, $12a^3 - 12ba^2 + 4b^2a - 4b^3$ by $4a^2 - 5ba + bb$; the Quotient is $3a$, and there remains $3baa + bba - 4b^3$, by which, according to the ſame Rules we ſhould divide $4a^2 - 5ba + bb$; but as the Diviſion of thoſe Quantities can not be performed without firſt preparing them, I obſerve 1°. that $b$ being common to all the Terms of the firſt Quantity, and not to thoſe of the ſecond, it cannot be a Part of the greateſt common Diviſor of thoſe Quantities, wherefore I take it away from all the Terms of this firſt Quantity, and I aſſume in its Place $3aa + ba + 4bb$. I obſerve 2°. that if the ſecond Quantity $4aa - 5ba + bb$ be multiplied by 3, which is no Diviſor of $3aa + ba - 4b^2$ the Diviſion will ſucceed; I divide therefore $12aa - 15ab + 3b^2$ by $3aa + ba - 4b^2$, the Quotient is 4, and there remains $-19ab + 19b^2$.

We have therefore no more to do than to find the greateſt common Diviſor of $3a^2 + ba - 4b^2$, and $-19ab + 19b^2$. As this Operation requires that we ſhould divide the firſt of thoſe two Quantities by the ſecond, and that it is neceſſary in order to make the Diviſion of the two firſt Terms ſucceed, to multiply the firſt by $19b$, which is an exact Diviſor of the ſecond, I take away this Diviſor from the ſecond, whereby it is reduced to $-a + b$.

But the greateſt common Diviſor of $3aa + ba - 4bb$ and $-a + b$, is $-a + b$ itſelf, ſince the Diviſion of thoſe two Quantities is performed exactly. Wherefore $-a + b$ is the greateſt common Diviſor of $3aa + ba - 4bb$ and of $-19ab + 19bb$, wherefore it is alſo the greateſt common Diviſor of $12aa - 15ab + 3bb$, and of $3aa + ba - 4b^2$, and conſequently of $4aa - 5ba + b^2$, and $3baa + bba - 4b^3$, as alſo of $12a^3 - 12baa + 4bba - 4b^3$, and $4aa - 5ba + bb$, and finally it is the greateſt common Diviſor of the propoſed Quantities $3a^3 - 3baa + bba - b^3$ and $4a^2 - 5ba + b^2$.

LXXIII.

It is eaſy to perceive that the greateſt common Diviſor of any other two Quantities may be found after the ſame Manner. The only Principle which we are obliged to add in this Enquiry to the Method of Art. LXXI. is that any two Quantities $A$ and $B$ will preſerve their greateſt common Diviſor when one of thoſe two Quantities, for Example, $A$, is multiplied or divided by a Quantity which has no common Diviſor with $B$.

General method for finding the greatest common divisor of specious or algebraic quantities.

The general Method for determining the greateſt common Diviſors, may be expreſſed thus. Let *A* and *B* be the two propoſed Quantities, firſt order thoſe two Quantities according to the ſame Letter, examine afterwards what Quantity *m* is moſt proper to multiply *A* by, in order that the Terms affected by the higheſt Power of the Letter according to which it is ordered may be diviſible by the Terms of *B* affected by the higheſt Power of the ſame Letter; if this multiplier *m* has no common Diviſor with *B*, multiply *A* by it, but if it has a common Diviſor *n*, take away this common Diviſor, both from *m* and *B*, and multiply *A* by $\frac{m}{n}$, which will form a new Quantity *C*, which aſſume in the Place of *A*. In like Manner aſſume in the Place of *B* the Quantity *D*, which reſults when *B* is divided by the Diviſor *n*, common with *m*.

Then divide *C* by *D*, and the Diviſion being performed, if it be exact, *D* will be the greateſt common Diviſor ſought of *A* and *B*, but if there be a Remainder *E*, perform upon *D* and *E* the ſame Operation as on *A* and *B*, and ſo on until two Quantities are obtained, of which one divides the other without a Remainder, and that Quantity will be the greateſt common Diviſor required.

It is proper to obſerve, that if before you attempt this Operation, you can diſcover in one of the propoſed Quantities *A* or *B* any Quantity which is an exact Diviſor of it and not of the other, it will be convenient to take away this Diviſor.

To render the Application of this Method familiar to Beginners, here follow more Examples.

LXXIV.

First example.

Let there be given the Quantities $q\,n\,p^3 + 3\,n\,p^2\,q^2 - 2\,n\,p\,q^3 - 2\,n\,q^4$, and $2\,m\,p^2\,q^2 - 4\,m\,p^4 - m\,p^3\,q + 3\,m\,p\,q^3$, of which we found $p - q$ to be a common Diviſor, only becauſe we knew before Hand that the firſt of thoſe two Quantities, divided by the ſecond ſhould give the ſame Quotient as the Quantity $-n\,q\,p^2 - 4\,p\,q^2\,n - 2\,n\,q^3$, divided by $4\,m\,p^3 + 5\,m\,p^2\,q + 3\,m\,p\,q^2$.

To reduce now thoſe two Quantities by the foregoing Method, I take away $q\,n$ which is common to all the Terms of the firſt of thoſe two Quantities, and is not contained in thoſe of the ſecond; I take away likewiſe $p\,m$, which is common to all the Terms of the ſecond without being contained in thoſe of the firſt, and thereby the Operation is reduced to find the greateſt common Diviſor of the Quantities *(A)* $-4\,p^3 - p^2\,q + 2\,p\,q^2 + 3\,q^3$ and *(B)* $p^3 + 3\,p\,p\,q - 2\,p\,q^2 - 2\,q^3$.

Dividing *A* by *B* the Quotient is $-4$, and there remains *(C)* $11\,p^2\,q - 6\,p\,q^2 - 5\,q^3$, as *B* ſhould be multiplied by $11\,q$ in order to render its firſt Term diviſible by the firſt Term of *C*, and that *q* is contained

in all the Terms of $C$, I multiply $B$ ſimply by 11, and I divide $C$ by $q$, hence the Queſtion is reduced to find the greateſt common Diviſor of the two Quantities $(D)$ $11\,p^3 + 33\,p^2\,q - 22\,p\,q^2 - 22\,q^3$, and $(E)$ $11\,p^2 - 6\,p\,q + 5\,q^2$.

I divide the firſt by the ſecond, the Quotient is $p$ and there remains $(F)$ $39\,p^2\,q - 17\,p\,q^2 - 22\,q^3$. As $(E)$ muſt be multiplied by $39\,q$ to render its firſt Term diviſible by that of this new Quantity $F$, and that $q$ is common to all the Terms of $F$, I multiply $E$ only by 39, and I divide its Product $(G)$ $429\,p^2 - 234\,p\,q - 195\,q^2$ by $(H)$ $39\,p^2 - 17\,p\,q - 22\,q^2$. The Quotient is 11, and the Remainder $(I)$ $-47\,p\,q + 47\,q^2$.

To render $H$ diviſible by $I$, we muſt multiply all its Terms by $47\,q$; but this Quantity is a Diviſor of $H$, wherefore I take it away from $H$, and there remains $q - p$ for Diviſor to $39\,p^2 - 17\,p\,q - 22\,q^2$. And as it divides it exactly, I conclude that $q - p$ is the greateſt common Diviſor ſought of the propoſed Quantities.

LXXV.

Second example.

Let there be given the two Quantities $a\,b + 2\,a\,a - 3\,b\,b - 4\,b\,c - a\,c - c\,c$ and $9\,a\,c + 2\,a\,a - 5\,a\,b + 4\,c\,c + 8\,b\,c - 12\,b\,b$, ranging thoſe two Quantities according to the Dimenſions of $a$, we have $2\,a\,a + b\,a - c\,a - 3\,b\,b - 4\,b\,c - c\,c$ and $2\,a\,a + 9\,c\,a - 5\,b\,a - 12\,b\,b + 8\,b\,c + 4\,c\,c$, or $(A)$ $2\,a\,a + (b - c) \times a - 3\,b\,b - 4\,b\,c - c\,c$ and $(B)$ $2\,a\,a + (9\,c - 5\,b) \times a - 12\,b\,b + 8\,b\,c + 4\,c\,c$.

Dividing the firſt by the ſecond, the Quotient is 1, and there remains $(C)$ $(6\,b - 10\,c) \times a + 9\,b\,b + 12\,b\,c - 5\,c\,c$. To render $B$ diviſible by this Quantity, I obſerve that it muſt be multiplied by $3\,b - 5\,c$, but before I perform this Operation, I try the Diviſion of $C$ by $3\,b - 5\,c$, which ſucceeds, and gives $(D)$ $2\,a + 3\,b + c$ for Quotient; the Queſtion is therefore reduced to find the greateſt common Diviſor of $B$ and $D$; but $B$ is diviſible exactly by $D$; wherefore $D$, or $2\,a + 3\,b + c$ is the greateſt common Diviſor ſought of $a\,b + 2\,a\,a - 3\,b\,b - 4\,b\,c - a\,c - c\,c$ and $9\,a\,c + 2\,a\,a - 5\,a\,b + 4\,c\,c + 8\,b\,c - 12\,b\,b$. The firſt of thoſe Quantities being the Product of $2\,a + 3\,b + c$ into $a - b - c$, the ſecond the Product of $2\,a + 3\,b + c$ into $a - 4\,b + 4\,c$, and thoſe two Quantities $a - b - c$, $a - 4\,b + 4\,c$ have no common Diviſor.

LXXVI.

Third example.

Let there be given the two Quantities $(A)$ $(d\,d - c\,c) \times a^2 + c^4 - d\,d\,c\,c$, and $(B)$ $4\,d\,a^2 - (2\,c\,c + 4\,c\,d) \times a + 2\,c^3$ ranged according to the Dimenſions of $a$. I firſt change $B$ into $(C)$ $2\,d\,a^2 - (c\,c + 2\,c\,d) \times a + c^3$ by taking away from all its Terms the Diviſor 2, which is not common with $A$. I afterwards multiply $A$ by

$2\,d$, in order to make the Diviſion ſucceed, the Quotient is $d\,d - c\,c$, and there remains (*D*) $(d\,d - c\,c) \times (c\,c + 2\,c\,d) \times a - (d\,d - c\,c) \times c^3 + 2\,d\,c^4 - 2\,d^3\,c\,c$. To make this Quantity ſerve as a Diviſor to *C*, we muſt firſt multiply *C* by $(d\,d - c\,c) \times (c\,c + 2\,c\,d)$.

But before I perform this Multiplication, I examine whether $(d\,d - c\,c) \times (c\,c + 2\,c\,d)$ be not a Diviſor or a Multiple of ſome Diviſor of *D*. By inveſtigating the greateſt common Diviſor of $(d\,d - c\,c) \times (c\,c + 2\,c\,d)$ and $-(d\,d - c\,c) \times c^3 + 2\,d\,c^4 - 2\,d^3\,c^2$, that is, of $d\,d\,c\,c - c^4 + 2\,c\,d^3 - 2\,c^3\,d$ and $-d\,d\,c^3 + c^5 + 2\,d\,c^4 - 2\,d^3\,c^2$; and I find that the ſecond of thoſe Quantities is the Product of the firſt into $-c$, and conſequently that the Quantity *D* is the Product of $(d\,d - c\,c) \times (c\,c + 2\,d\,c)$ into $a - c$; wherefore inſtead of multiplying *C* by $(d\,d - c\,c) \times (c\,c + 2\,d\,c)$. I divide *D* by this Quantity, and the Quotient is (*E*) $a - c$, and as $a - c$ divides *C* exactly, I conclude, that $a - c$ is the greateſt common Diviſor ſought.

LXXVII.

Another manner of finding the greateſt common diviſor of the quantities in the foregoing example.

The greateſt common Diviſor of two Quantities may ſometimes be obtained without having recourſe to the general Method. For Example, the two foregoing Quantities $(d\,d - c\,c) \times a\,a + c^4 - d\,d\,c\,c$ and $4\,d\,a\,a - (2\,c\,c + 4\,c\,d) \times a + 2\,c^3$ being ordered according to *d*, and conſequently being reduced to this Form $(a\,a - c\,c) \times d\,d + c^4 - a\,a\,c\,c$ and $(4\,a\,a - 4\,a\,c) \times d + 2\,c^3 - 2\,c^2\,a$, it is eaſy to perceive that $a\,a - c\,c$ is a Diviſor of the firſt, and $c - a$ a Diviſor of the ſecond. But $a^2 - c^2$ is diviſible by $c - a$, wherefore $c - a$ is a Diviſor of the two propoſed Quantities; I divide therefore both one and the other by $c - a$, and the Quotients are $(c\,c - d\,d) \times (c + a)$; and $4\,a\,d + 2\,c\,c$; which by Inſpection are found to have no common Diviſor, conſequently $c - a$ or $a - c$ is the greateſt common Diviſor of the propoſed Quantities.

LXXVIII.

Other quantities whoſe greateſt common diviſor is found independent of the foregoing method.

Let it be propoſed to find the greateſt common Diviſor of the two Quantities $6\,a^5 + 15\,a^4\,b - 4\,a^3\,c^2 - 10\,a\,a\,b\,c\,c$ and $9\,a^3\,b - 27\,a\,a\,b\,c - 6\,a\,b\,c\,c + 18\,b\,c^3$, I firſt take away $a\,a$ from all the Terms of the firſt, and $3\,b$ from all thoſe of the ſecond, whereby they are reduced to $6\,a^3 + 15\,a^2\,b - 4\,a\,c\,c - 10\,b\,c\,c$ and $3\,a^3 - 9\,a\,a\,c - 2\,a\,c\,c + 6\,c^3$; but as *b* is not contained in any of the Terms of the ſecond Quantity, I conclude, that if it has a common Diviſor with the firſt, it muſt have one with its two Parts $6\,a^3 - 4\,a\,c\,c$ and $15\,a^2\,b - 10\,b\,c\,c$, and that thoſe two Parts ſhould alſo have the ſame common Diviſor. But it appears by Inſpection that $3\,a\,a - 2\,c\,c$ is the common Diviſor of thoſe two Parts, wherefore it is the greateſt common Diviſor of the propoſed Quantities, if they have one, and dividing, in effect, thoſe two

Quantities by it, the Diviſion ſucceeds, conſequently it is their greateſt common Diviſor.

LXXIX.

From what precedes, it is eaſy to conclude, that when two Quantities are required in a Problem, there muſt be two Equations given. Likewiſe when there are three unknown Quantities there muſt be three Equations in order to determine them, and in general there muſt be as many Equations as Quantities required. As to the Manner of diſengaging the unknown Quantities involved in thoſe Equations, it is preciſely the ſame as for Equations involving two unknown Quantities. For let three Equations be given, each involving three unknown Quantities $x$, $y$, $z$; if a Value of $x$ be deduced from each of thoſe Equations, and thoſe different Values be put equal to one another, it is obvious there will reſult two new Equations, involving only $y$ and $z$, which will be reſolved by the foregoing Method. If four Equations are given, involving four unknown Quantities, their Values may be found after the ſame Manner, and ſo on.

When there are three quantities required in a problem, there muſt be three equations given to find them.

How the values of the unknown quantities involved in thoſe equations are found.

The Uſe of this Method will be more clearly underſtood by Help of the following Problem, containing the greateſt Complication that Equations of the firſt Degree involving three unknown Quantities are ſuſceptible of.

LXXX.

*The Prices of three Magazines, each containing three Sorts of Grain, and the Number of Meaſures which each Magazine contains of thoſe three different Sorts of Grain being given; to determine the Price of a Meaſure of each Sort of Grain.*

Problem in which three unknown quantities are required.

Let $a$, $b$, $c$ expreſs the Numbers of Meaſures of each Sort of Grain contained in the firſt Magazine, and let $m$ be the Price of this Magazine.

Let $d$, $e$, $f$ expreſs the Numbers of Meaſures of each Sort of the ſame Kinds of Grain, contained in the ſecond Magazine, and let $n$ be the Price of this Magazine.

Let $g$, $h$, $k$ expreſs the Numbers of Meaſures of each Sort of the ſame Kinds of Grain contained in the third Magazine, and let $p$ be the Price of this Magazine.

And let $x$, $y$, $z$ expreſs the Prices of a Meaſure of each Sort of Grain.

It is manifeſt that the Price of the Quantity of the firſt Sort of Grain contained in the Magazine $m$ will be expreſſed by $a\,x$, ſince $a$ is the Number of Meaſures of this Sort of Grain, and $x$ the Price of a Meaſure; in like Manner the Price of the Quantity of the ſecond Sort of Grain contained in the ſame Magazine, will be expreſſed by $b\,y$, and the Price of the Quantity of the third Sort of Grain contained in the ſame Magazine, will be expreſſed by $c\,z$. Wherefore ſince the Price

$m$ of this Magazine is equal to thoſe three Sums together, we will have $ax + by + cz = m$.

Expreſſing in like Manner the Conditions reſpecting the two other Magazines, we will have $dx + ey + fz = n$ and $gx + by + kz = p$.

It is now Queſtion to deduce from thoſe Equations the Values of $x, y, z$; with this View I firſt deduce the Value of $x$ from the firſt Equation, which is $\frac{m - by - cz}{a}$, and putting this Value of $x$ equal to that deduced from the ſecond, we will have the Equation $\frac{m - by - cz}{a} = \frac{n - ey - fz}{d}$, putting afterwards the ſame Value $\frac{m - by - cz}{a}$ equal to that deduced from the third Equation, we will have $\frac{m - by - cz}{a} = \frac{p - by - kz}{g}$.

From the firſt of thoſe two Equations I deduce $an - dm = aey - dby + afz - dcz$, or $y = \frac{an - dm + dcz - afz}{ae - db}$, from the ſecond I deduce $ap - mg = aby + akz - bgy - cgz$, or $y = \frac{ap - gm + gcz - akz}{ab - gb}$.

Putting thoſe two Values of $y$ equal, it is manifeſt that we will have an Equation involving no other unknown Quantity but $z$, and that ſolving this Equation the Value of $z$ will be found. As the Calculations in this Operation would be conſiderable, I ſhall ſhew how they may be avoided by employing ſome Abbreviations which the firſt Analyſts who were engaged in great Calculations eaſily imagined.

LXXXI.

Manner of abridging the calculations by particular denominations.

Thoſe Abbreviations conſiſt in ſubſtituting new Letters in the Place of ſeveral Terms compoſed of known Quantities.

Inſtead of $an - dm$ I ſubſtitute $A$, for $dc - af$, $B$, for $ae - db$, $C$, for $ap - gm$, $D$, for $gc - ak$, $E$, for $ab - gb$, $F$.

By thoſe new Denominations the foregoing Equations will beccome $y = \frac{A + Bz}{C}$ and $y = \frac{D + Ez}{F}$, which gives $AF + BFz = DC + CEz$, from whence is deduced $z = \frac{DC - AF}{BF - CE} = \frac{aep - abn + dbm - dbp + gbn - gem}{aek - abf + dbc - dbk + gbf - gcc} = \ldots\ldots$ $\frac{(ae - db)p + (gb - ab)n + (db - ge)m}{(ae - db)k + (gb - ab)f + (db - ge)c}$, ſubſtituting afterwards this Value of $z$ in one of the two foregoing Values of $y$, in

the first, for Example, we will have $y = \frac{A + \frac{BDC - BAF}{BF - CE}}{C}$, or

$$y = \frac{BD - AE}{BF - CE} = \frac{akn - afp + dcp - dkm + gfm - gcn}{aek - abf + dbc - dbk + gbf - gec} = \frac{(ak - gc)\,n + (dc - af)\,p + (gf - dk)\,m}{(ae - db)\,k + (gb - ab)\,f + (db - ge)\,c}.$$

This being done, I substitute those Values of $y$ and $z$ in one of the foregoing Values of $x$, in $\frac{m - cz - by}{a}$, for Example, and there results $x = \frac{m}{a} - \frac{c}{a} \times \left(\frac{DC - AF}{BF - CE}\right) - \frac{b}{a} \times \frac{BD - AE}{BF - CE}$,

or $x = \frac{m \times (BF - CE) - c \times (DC - AF) - b \times (BD - AE)}{a \times (BF - CE)} = \frac{(ek - fh)\,m + (cb - bk)\,n + (bf - ce)\,p}{(ae - db)\,k + (gb - ab)\,f + (db - ge)\,c}$.

LXXXII.

To apply this Method to an Example, let us suppose that the first Magazine contains 30 Measures of Rye, 20 of Barley, 10 of Wheat, and that it cost £11 10 *s.*

That the second contains 15 Measures of Rye, 6 of Barley, and 12 of Wheat, and that it cost £6 18 *s.*

That the third Magazine contains 10 Measures of Rye, 5 of Barley, and 4 of Wheat, and that it cost £3 15 *s.* To determine the Prices of a Measure of Rye, that of Barley, and that of Wheat, we must put $a = 30$, $b = 20$, $c = 10$, $m = 230$, $d = 15$, $e = 6$, $f = 12$, $n = 138$, $g = 10$, $b = 5$, $k = 4$, $p = 75$.

Substituting those Values in the Formulas, you will find $x = 4$, $y = 3$, and $z = 5$, consequently the Price of a Measure of Rye is 4 Shillings, that of a Measure of Barley 3 Shillings, and that of a Measure of Wheat 5 Shillings.

LXXXIII.

All problems of the first degree in which three quantities are required when reduced to equations are contained in the foregoing problem.

As the Equations of the foregoing Problem are the most general of the first Degree involving three unknown Quantities, since each involves three unknown Quantities combined with known Quantities, it follows that every Problem of the first Degree involving three unknown Quantities, is included in the foregoing, as soon as it is expressed analytically. To give an Example, let the following Problem be proposed.

There are three Ingots composed of different Metals melted down together, of the first of which a Pound (Averdupois) contains, of Silver 7 Ounces, of Brass 3 Ounces, and of Tin 6 Ounces; of the second a Pound con-

tains, of Silver 12 Ounces, of Braſs 3 Ounces, of Tin 1 Ounce; and a Pound of the third contains, of Silver 4 Ounces, of Braſs 7 Ounces, and of Tin 5 Ounces: How much of each Ingot muſt be taken to make a fourth, which ſhall contain, of Silver 8 Ounces, of Braſs $3\frac{3}{4}$ Ounces, and of Tin $4\frac{1}{4}$ Ounces.

Let $x$, $y$, $z$ expreſs the Number of Ounces to be taken of each of thoſe Ingots.

It is manifeſt that $\frac{7}{16}x$ will expreſs the Quantity of Silver in the Portion of the firſt Ingot, that $\frac{12}{16}y$ will expreſs what is contained in the Portion of the ſecond Ingot, and that $\frac{4}{16}z$ that contained in the Portion of the third.

As the Sum of thoſe three Quantities ſhould be 8 Ounces of Silver, there reſults the Equation $\frac{7}{16}x + \frac{12}{16}y + \frac{4}{16}z = 8$, or $7x + 12y + 4z = 128$.

In like Manner the Quantity of Braſs taken out of each of the three Ingots, will be expreſſed by $\frac{3}{16}x$ $\frac{3}{16}y$, and $\frac{7}{16}z$ the Sum of which ſhould be $3\frac{3}{4}$ Ounces, wherefore $\frac{3}{16}x + \frac{3}{16}y + \frac{7}{16}z = \frac{15}{4}$, or $3x + 3y + 7z = 60$.

The Quantity of Tin taken of each Ingot, will, in like Manner be expreſſed by $\frac{6}{16}x$, $\frac{1}{16}y$, $\frac{5}{16}z$, the Sum of which ſhould be $4\frac{1}{4}$ Ounces, wherefore $\frac{6}{16}x + \frac{1}{16}y + \frac{5}{16}z = 4 + \frac{1}{4}$, or $6x + y + 5z = 68$.

We have no more to do now than to reſolve thoſe three Equations, which will be effected by putting in the foregoing Solution $a = 7$, $b = 12$, $c = 4$, $m = 128$, $d = 3$, $e = 3$, $f = 7$, $n = 60$, $g = 6$, $h = 1$, $k = 5$, $p = 68$.

Subſtituting thoſe Values in the Formulas, you will find $x = 8$, $y = 5$ and $z = 3$, that is, we muſt take 8 Ounces of the firſt Ingot, 5 Ounces of the ſecond, and 3 of the third to form the Ingot required.

### LXXXIV.

In what caſes problems are indeterminate.

It is eaſy to perceive that if there are more Quantities required than Equations given, the Queſtion is not limited to determinate Quantities, but is capable of a Number of Solutions, which however are confined within certain Limits, as will appear by the following Example.

Firſt example.

*There are three Ingots of Gold: The Mark * of the firſt contains* 23 *Carats of fine Gold, that of the ſecond* 21, *and that of the third* 18. *How much of each Ingot muſt be taken to compoſe a fourth, weighing* 9 *Marks, each Mark containing* 22 *Carats of fine Gold.*

Let $x$, $y$, $z$ expreſs the Number of Marks to be taken of each of the Ingots.

It is manifeſt that $23x$ will expreſs the Quantity of pure Gold in the Portion of the firſt Ingot, $21y$ that in the Portion of the ſecond, and $18z$ that in the Portion of the third. Adding therefore thoſe three

* The Mark contains 8 Ounces. The Carat is the $\frac{1}{24}$ of a Mark.

Quantities, their Sum ſhould contain 9 Times 22 Carats, conſequently $23x + 21y + 18z = 22 \times 9$. And as the fourth Ingot compoſed of thoſe three Portions ſhould weigh 9 Marks, $x + y + z = 9$.

All the Conditions of the Problem being expreſſed, it is manifeſt that the Queſtion is not limited to determinate Quantities, becauſe there are three unknown Quantities and only two Equations, but it is evident that if one of the three unknown Quantities be determined, the other two will be alſo determined, and the Suppoſitions which may be made and the Limits within which they are confined, will be diſcovered by ſolving thoſe Equations; which will be effected by putting $a = 23$, $b = 21$, $c = 18$, $m = 22 \times 9$, $d = 1$, $e = 1$, $f = 1$, $n = 9$.

For then the Equation $an - md = aey - bdy + afz - cdz$ is reduced to $an - m = (a - b) \times y + (a - c) \times z$,

or $y = \frac{an - m - (a - c)z}{a - b}$, where I obſerve, 1°. that the Product $(a - c)z$ of the greateſt Difference $a - c$ into $z$ cannot exceed the Product $an - m$ of the Number of Marks into the leaſt Difference, otherwiſe $y$ would become negative. I obſerve, 2°. that $z$ cannot be ſuppoſed $= \frac{an - m}{a - c}$, becauſe $y$ would in this Caſe become $= 0$, that is, the fourth Ingot would contain none of the ſecond, which is contrary to the Conditions of the Problem, hence the Limits within which the Values of $z$ are confined, are $z < 1 + \frac{2}{3}$ and $z > 0$. The Sign $>$ being employed by the Analyſts to denote the Inequality of the two Quantities between which it is placed, the Point being always directed towards the leaſt Quantity.

It is manifeſt that all the Suppoſitions for $z$ included within thoſe Limits will ſolve the Problem.

Let, for Example, $z = 1 + \frac{2}{3}$ Marks, then $y = \frac{9 - 5 \times \frac{5}{3}}{2} = \frac{9 - 8}{2} = \frac{1}{2}$; which denotes that if we take $1 + \frac{2}{3}$ Marks of the third Ingot, we muſt take $\frac{1}{2}$ a Mark of the ſecond, and conſequently $6\frac{9}{10}$ of the firſt. In effect $1 + \frac{2}{3}$ Marks of the third Ingot contains $28\frac{1}{3}$ Carats of fine Gold, $\frac{1}{2}$ a Mark of the ſecond contains $10\frac{1}{2}$ Carats, and $6\frac{9}{10}$ Marks of the firſt contains $158\frac{7}{10}$ Carats, and conſequently the three Portions together contain 198 Carats of pure Gold, the ſame Quantity that is contained in 9 Marks, each Mark conſiſting of 22 Carats of fine Gold.

If we ſuppoſe $z = 1$, then $y = 2$, which ſhews that if we take 1 Mark of the third Ingot we muſt take 2 of the ſecond, and conſequently 6 of the firſt. In effect 1 Mark of the third furniſhes 18 Carats, 2 Marks of the ſecond will give 42, and 6 Marks of the firſt makes 138,

which together amount to 198 Carats, which is preciſely the ſame Quantity, as is contained in 9 Marks, each Mark conſiſting of 22 Carats of fine Gold.

LXXXV.

To accuſtom Beginners to the Method of limiting the Anſwers to all Sorts of Queſtions of this Kind, here follows another Example.

Second example.

Suppoſe a Piece of Metal to be compoſed of Tin, Gold and Copper, melted down together, its weight to be 375 Ounces, its Bulk 80 cubick Inches, let the Weight of a cubick Inch of Tin be $4\frac{3}{8}$ Ounces, that of Gold $11\frac{5}{8}$ Ounces, and that of Copper $5\frac{1}{2}$ Ounces, it is required to determine how much of each of thoſe Metals are contained in the propoſed Maſs.

Let $x$, $y$, $z$ expreſs the Quantities of each of thoſe Metals.

As the Quantity of Tin added to that of Gold and Copper ſhould compoſe the whole Bulk, $x + y + z = 80$. And it being manifeſt that $4\frac{3}{8}x$ expreſſes the Weight of the Quantity of Tin in the Mixture, $11\frac{5}{8}y$ that of the Gold, and $5\frac{1}{2}z$ that of the Copper, and that thoſe three Weights together ſhould be equal to the whole Weight of the Mixture $4\frac{3}{8}x + 11\frac{5}{8}y + 5\frac{1}{2}z = 375$ or taking away the Fractions $35x + 93y + 41z = 3000$.

Thoſe two Equations expreſſing the Conditions of the Problem, it is manifeſt that it is not limited to determinate Quantities, but if one of the unknown Quantities be determined, the two others will be determined alſo. In order therefore to ſolve thoſe two Equations, and thereby diſcover what Suppoſitions may be made, and within what Limits they are confined, I put $a = 1$, $b = 1$, $c = 1$, $m = 80$, $d = 35$, $e = 93$, $f = 41$, $n = 3000$, ſubſtituting thoſe Values in the Equation $an - dm = (ae - bd)y + (af - cd) \times z$, there reſults $58y + 6z = 200$.

From whence it appears that $z < 33\frac{1}{3}$, that is, that there cannot be $33\frac{1}{3}$ cubick Inches of Copper in the Mixture, for the Equation $58y + 6z = 200$, becoming in this Caſe $58y + 6 \times 33\frac{1}{3} = 200$ will be changed into $58y + 200 = 200$, from whence is deduced $58y = 200 - 200 = 0$, that is, there would be no Gold in the Mixture, which is contrary to the Suppoſition, but all the Suppoſitions for the Value of $z$ leſs than $33\frac{1}{3}$ will anſwer the Conditions of the Problem.

Let us ſuppoſe $z = 4$, conſequently $6z = 24$, and $58y = 176$, or $y = 3\frac{1}{29}$, which denotes that if the Mixture be ſuppoſed to contain 4 cubick Inches of Copper, it will contain $3\frac{1}{29}$ cubick Inches of Gold, conſequently $72\frac{28}{29}$ cubick Inches of Tin, ſince thoſe three Quantities ſhould compoſe 80 cubick Inches, and $4\frac{3}{8} \times 72\frac{28}{29} + 11\frac{5}{8} \times 3\frac{1}{29} + 5\frac{1}{2} \times 4 = 375$ Ounces.

If we ſuppoſe $z = 3$, then $6z = 18$, and the Equation $58y + 6z = 200$ will be changed into $58y + 18 = 200$, wherefore $y = \frac{182}{58} = 3\frac{4}{29}$, which denotes that if the Mixture is ſuppoſed to contain 3 cubick Inches of Copper, it will contain $3\frac{4}{29}$ cubick Inches of Gold, and conſequently $73\frac{25}{29}$ cubick Inches of Tin. Thoſe three Quantities together making 80 cubick Inches, and $4\frac{3}{8} \times 73\frac{25}{29} + 11\frac{1}{8} \times 3\frac{4}{29} + 5\frac{1}{8} \times 3 = 375$ Ounces.

LXXXVI.

Inconveniency to which the foregoing method for exterminating unknown quantities is liable.

It is obvious that when the Number of Equations and unknown Quantities is conſiderable, the Calculation for finding their Values according to the foregoing Method would be very laborious, we ſhall therefore proceed to explain what Means the Analyſts have found to remedy this Inconveniency.

Obſervations which have ſerved to improve this method.

On examining the Formulas expreſſing the Values of the unknown Quantities in the general Solutions of the Problems of Art. LVI and LXXX, it will appear, that the common Denominator of thoſe Values is formed of all the Products that can be made of a Number of Coeficients equal to the Number of Equations, and that are taken each from a different Equation, and are prefixed to a different unknown Quantity; thoſe which involve the Products of two ſuch Coeficients having contrary Signs.

For Example, in two Equations involving two unknown Quantities $ax + by = c$ and $dx + ey = f$, all the Products that can be made of two Coeficients taken each from a different Equation, and are prefixed to a different unknown Quantity, are $ae$, $bd$, to which if contrary Signs be given, there will reſult $ae - bd$, which is the common Denominator of the Values of $x$ and $y$ deduced from thoſe Equations Art. LVI. In like Manner in three Equations including three unknown Quantities $ax + by + cz = m$, $dx + ey + fz = n$, and $gx + by + kz = p$, all the Products that can be made of three Coeficients taken each from a different Equation, and are prefixed to a different unknown Quantity, are $aek$, $abf$, $dbc$, $dbk$, $gbf$, $gec$, and if contrary Signs be given to thoſe which involve the Products of two ſuch Coeficients, there will reſult $aek - abf + dbc - dbk + gbf - gec$, which is the common Denominator of the Values of $x$, $y$ and $z$, deduced from thoſe Equations Art. LXXX.

It alſo appears that the common Denominator is changed into the Numerator of any of thoſe Values, by ſubſtituting in this Denominator in the Place of the Coeficients of the unknown Quantity in the given Equations thoſe which affect no unknown Quantity.

For Example, if we ſubſtitute in the common Denominator $ae - db$, for the Coeficients $e$ and $b$ of $y$, in the Equations $ax + by = c$, $dx + ey = f$, thoſe $f$, $c$ which affect no unknown Quantity, there will

result $af - dc$, which is the Numerator of the Value of $y$, in like Manner ſubſtituting in this Denominator for the Coeficients $a$ and $d$ of $x$, thoſe $c$ and $f$, which affect no unknown Quantity, there will reſult $ce - bf$, which is the Numerator of the Value of $x$.

If we ſubſtitute in the common Denominator $aek - abf + dbc - dbk + gbf - gec$ for the Coeficients $k$, $f$, $c$ of $z$ in the Equations $ax + by + cz = m$, $dx + ey + fz = n$, $gx + by + kz = p$, thoſe $p$, $n$, $m$ which affect no unknown Quantity, there will reſult $aep - abn + dbm - dbp + gbn - gem$, which is the Numerator of the Value of $z$.

In like Manner ſubſtituting in this Denominator for the Coeficients $b$, $e$, $b$ of $y$, thoſe $p$, $n$, $m$, there will reſult $akn - afp + dcp - dkm + gfm - gcn$, which is the Numerator of the Value of $y$.

Finally ſubſtituting in this Denominator for the Coeficients $g$, $d$, $a$ of $x$, thoſe $p$, $n$, $m$, there will reſult $ekm - fbm + cbn - bkn + bfp - cep$, which is the Numerator of the Value of $x$.

LXXXVII.

General rule for finding the values of any number of quantities required when as many ſimple equations are given.

From theſe Obſervations the Analyſts have deduced this general Rule, for exterminating unknown Quantities in Equations of the firſt Degree.

Let $a$, $b$, $c$, $d$, &c. be the Coeficients of thoſe unknown Quantities and $A$ that which affects no unknown Quantity, in the firſt Equation.

Let $a'$, $b'$, $c'$, $d'$, &c. be the Coeficients of the ſame unknown Quantities, and $A'$ that which affects no unknown Quantity, in the ſecond Equation.

Let $a''$, $b''$, $c''$, $d''$, &c. be the Coeficients of the ſame unknown Quantities, and $A''$ that which affects no unknown Quantity in the third, and ſo on.

Form the two Permutations $ab$ and $ba$ and write down $ab - ba$; with thoſe two Permutations and the Letter $c$, form all the poſſible Permutations, obſerving to change the Sign as often as $c$ will change its Place in $ab$, as likewiſe in $ba$; and there will reſult

$$abc - acb + cab - bac + bca - cba.$$

With thoſe ſix Permutations and the Letter $d$, form all the Permutations poſſible, obſerving to change the Sign as often as $d$ will change its Place in the ſame Term, and there will reſult

$$abcd - abdc + adbc - dabc - acbd + acdb - adcb + dacb + cabd - cadb + cdab - dcab - bacd + badc - bdac + dbac + bcad - bcda + bdca - dbca - cbad + cbda - cdba + dcba$$

and proceed in this Manner until all the Coeficients of the firſt Equation are exhauſted.

Afterwards preſerve the Letters which occupy the firſt Place, give thoſe which occupy the ſecond Place the ſame Mark they have in the ſecond Equation, and thoſe which occupy the third Place the ſame Mark they have in the third Equation, and ſo on. And this Reſult will be the common Denominator of the Values of the unknown Quantities.

The common Denominator being thus formed, the Value of $x$ will be obtained by giving to this Denominator the Numerator which is found by changing in all its Terms $a$ into $A$, and the Value of $y$ is the Fraction which has the ſame Denominator, and for Numerator the Quantity which reſults by changing $b$ into $A$ in all the Terms of the Denominator, and in like Manner the Values of the other unknown Quantities will be obtained.

Hence if there be two Equations and two unknown Quantities $a\,x + b\,y = A$, $a'\,x + b'\,y = A'$, the common Denominator will be $a\,b' - b\,a'$ or $a\,b' - a'\,b$.

If there be three Equations and three unknown Quantities $a\,x + b\,y + c\,z = A$, $a'\,x + b'\,y + c'\,z = A'$, $a''\,x + b''\,y + c''\,z = A''$, the common Denominator will be $a\,b'\,c'' - a\,c'\,b'' + c\,a'\,b'' - b\,a'\,c'' + b\,c'\,a'' - c\,b'\,a''$, or $a\,b'\,c'' - a\,b''\,c' + a'\,b''\,c - a'\,b\,c'' + a''\,b\,c' - a''\,b'\,c$, or $(a\,b' - a'\,b)\,c'' + (a''\,b - a\,b'')\,c' + (a'\,b'' - a''\,b')\,c$.

If there be four Equations and four unknown Quantities $a\,x + b\,y + c\,z + d\,t = A$, $a'\,x + b'\,y + c'\,z + d'\,t = A'$, $a''\,x + b''\,y + c''\,z + d''\,t = A''$, $a'''\,x + b'''\,y + c'''\,z + d'''\,t = A'''$, the common Denominator will be found after ranging the Letters in alphabetic Order to be

$$\begin{aligned}
&a\,b'\,c''\,d''' - a\,b'\,c'''\,d'' + a\,b''\,c'''\,d' - a\,b''\,c'''\,d - a\,b''\,c'\,d''' \\
&+ a\,b'''\,c'\,d'' - a\,b'''\,c''\,d' + a'\,b'''\,c''\,d - a'\,b''\,c\,d''' - a'\,b'''\,c\,d'' \\
&+ a''\,b'''\,c\,d' - a''\,b'''\,c'\,d - a'\,b\,c''\,d''' + a'\,b\,c'''\,d'' - a'\,b\,c'''\,d' \\
&+ a''\,b'\,c'''\,d + a''\,b\,c'\,d''' - a'''\,b\,c'\,d'' + a'''\,b\,c''\,d' - a'''\,b'\,c''\,d \\
&- a''\,b'\,c\,d''' - a'''\,b'\,c\,d'' - a''\,b''\,c\,d' + a'''\,b''\,c'\,d
\end{aligned}$$

$$\begin{aligned}
\text{or } &[(a\,b' - a'\,b)\,c'' + (a''\,b - a\,b'')\,c' + (a'\,b'' - a''\,b')\,c]\,d''' \\
&+[(a'\,b - a\,b')\,c''' + (a\,b''' - a'''\,b)\,c' + (a'''\,b' - a'\,b''')\,c]\,d'' \\
&+[(a'''\,b - a\,b''')\,c'' + (a\,b'' - a''\,b)\,c''' + (a''\,b''' - a'''\,b'')\,c]\,d' \\
&+[(a'\,b''' - a'''\,b')\,c'' + (a'''\,b'' - a''\,b''')\,c' + (a''\,b' - a'\,b'')\,c''']\,d
\end{aligned}$$

LXXXVIII.

Now it is eaſy to obſerve, 1°. that the firſt Term of any one of thoſe Denominators, is formed of the foregoing Denominator multiplied by the next Letter in the Order of the Alphabet, which it does not include, this Letter being affected by the Mark which immediately follows the higheſt of thoſe in this ſame Denominator. Obſervations which have ſerved to render this rule more ſimple

2°. That the ſecond Term is formed of the firſt, by changing the higheſt Mark in it into that which is immediately below it, and that which is immediately below the higheſt into the higheſt, as alſo by changing the Signs.

3°. The third Term is formed of the firſt, by changing in it the higheſt Mark into that of two Numbers below it, and that of two Numbers below the higheſt into the higheſt, as alſo by changing the Signs.

4°. The fourth is formed of the firſt by changing the higheſt Mark in it, into that which is three Marks below it, and reciprocally, as alſo by changing the Signs.

For Example, the ſecond Denominator has for firſt Term $(a\,b' - a'\,b)\,c''$ which is the firſt Denominator multiplied by $c''$, which is the Letter which follows immediately the Letters $a$ and $b$, and which has the Mark $''$, which follows immediately the Mark $'$, the higheſt of thoſe which enter in the Expreſſion $a\,b' - a'\,b$.

The ſecond Term of this ſecond Denominator is $(a''\,b - a\,b'')\,c'$, which is no other than $(a\,b' - a'\,b)\,c''$, in which the Signs have been changed, and the higheſt Mark $''$ into $'$ which is immediately below it, and $'$ which is immediately below the higheſt into the higheſt $''$, and ſo on.

Hence the common Denominator of the Values of the unknown Quantities in any Number of Equations is eaſily determined, for Example, if there be five Equations and five unknown Quantities, the common Denominator will be found by this Method to be

$$\left.\begin{array}{l}
[(a\,b' - a'\,b)\,c'' + (a''\,b - a\,b'')\,c' + (a'\,b'' - a''\,b')\,c]\,d''' \\
+[(a'\,b - a\,b')\,c''' + (a\,b''' - a'''\,b)\,c' + (a'''\,b' - a'\,b''')\,c]\,d'' \\
+[(a'''\,b - a\,b''')\,c'' + (a\,b'' - a''\,b)\,c''' + (a''\,b''' - a'''\,b'')\,c]\,d' \\
+[(a'\,b''' - a'''\,b')\,c'' + (a''' b'' - a''\,b''')\,c' + (a''\,b' - a'\,b'')\,c''']\,d
\end{array}\right\} e''''$$

$$\left.\begin{array}{l}
+[(a'\,b - a\,b')\,c'' + (a\,b'' - a''\,b)\,c' + (a''\,b' - a'\,b'')\,c]\,d'''' \\
+[(a\,b' - a'\,b)\,c''' + (a'''\,b - a\,b''')\,c' + (a'\,b''' - a'''\,b')\,c]\,d''' \\
+[(a\,b'''' - a''''\,b)\,c'' + (a''\,b - a\,b'')\,c''' + (a''''\,b'' - a''\,b'''')\,c]\,d'' \\
+[(a''''\,b' - a'\,b'''')\,c'' + (a''\,b'''' - a''''\,b'')\,c' + (a'\,b'' - a''\,b')\,c'''']\,d
\end{array}\right\} e'''$$

$$\left.\begin{array}{l}
+[(a'\,b - a\,b')\,c''' + (a\,b''' - a'''\,b)\,c' + (a'''\,b' - a'\,b''')\,c]\,d'''' \\
+[(a\,b' - a'\,b)\,c'''' + (a''''\,b - a\,b'''')\,c' + (a'\,b'''' - a''''\,b')\,c]\,d''' \\
+[(a\,b''' - a'''\,b)\,c'''' + (a''''\,b - a\,b'''')\,c''' + (a'''\,b'''' - a''''\,b''')\,c]\,d'' \\
+[(a'''\,b' - a'\,b''')\,c'''' + (a''''\,b''' - a'''\,b'''')\,c' + (a'\,b'''' - a''''\,b')\,c''']\,d
\end{array}\right\} e''$$

$$\left.\begin{array}{l}
+[(a''''\,b - a\,b'''')\,c'' + (a\,b'' - a''\,b)\,c''' + (a''\,b'''' - a''''\,b'')\,c]\,d' \\
+[(a\,b'''' - a''''\,b)\,c''' + (a'''\,b - a\,b''')\,c'''' + (a''''\,b''' - a'''\,b'''')\,c]\,d'' \\
+[(a\,b''' - a'''\,b)\,c'' + (a''\,b - a\,b'')\,c''' + (a'''\,b'' - a''\,b''')\,c]\,d'''' \\
+[(a'''\,b'''' - a''''\,b''')\,c'' + (a''\,b''' - a'''\,b'')\,c'''' + (a''''\,b'' - a''\,b'''')\,c''']\,d
\end{array}\right\} e'$$

$$\left.\begin{array}{l}
+[(a'\,b'''' - a''''\,b')\,c'' + (a''''\,b'' - a''\,b'''')\,c' + (a''\,b' - a'\,b'')\,c''''']\,d''' \\
+[(a''''\,b' - a'\,b'''')\,c''' + (a'''\,b'''' - a''''\,b''')\,c' + (a'\,b''' - a'''\,b')\,c'''']\,d'' \\
+[(a''''\,b''' - a'''\,b'''')\,c'' + (a''\,b'''' - a''''\,b'')\,c''' + (a'''\,b'' - a''\,b''')\,c'''']\,d' \\
+[(a'''\,b'' - a''\,b''')\,c'' + (a''\,b''' - a'''\,b'')\,c' + (a'\,b'' - a''\,b')\,c''']\,d''''
\end{array}\right\} e$$

LXXXIX.

When there are as many ſimple Equations given as Quantities required, generally ſpeaking, the Problem is limited, but in ſome particular Caſes it may be unlimited or indeterminate, and in others impoſſible, this happens when the common Denominator is found equal to nothing; that is, if there be two Equations given, when $a b' - a' b = 0$, if there be three, when $(a b' - a' b) c'' + (a'' b - a b'') c' + (a' b'' - a'' b') c = 0$, &c. then if the Quantities $A'$, $A'$, $A''$, $A'''$, &c. are ſuch that the Numerators are alſo equal to nothing, the Problem is indeterminate; becauſe the Fractions $\frac{0}{0}$ which ſhould expreſs the Values of the unknown Quantities are indeterminate; but if the Quantities $A$, $A'$, $A''$, &c. are ſuch that the common Denominator being equal to nothing, the Numerators or any one of them is not equal to nothing, the Problem is impoſſible, or at leaſt the Quantities required are all or ſome of them greater than any aſſignable Quantities. For Example, let there be given the two Equations $2 = 3x - 2y$ and $5 = 6x - 4y$, there will reſult $x = \frac{2}{0}$ and $y = \frac{3}{0}$. Now as 0 is leſs than any aſſignable Quantity, it follows that $x$ and $y$ are greater than any aſſignable Quantities.

Problems which furniſh as many equations as quantities required are not always limited, but in ſome caſes are unlimited and in others impoſſible.

If the unknown Quantities be diſengaged according to the ordinary Method, there would reſult this abſurd Equation $\frac{2}{3} = \frac{5}{6}$, for the firſt Equation gives $x = \frac{2}{3}y + \frac{2}{3}$, and the ſecond $x = \frac{4}{6}y + \frac{5}{6}$. Wherefore $\frac{2}{3}y + \frac{2}{3} = \frac{4}{6}y + \frac{5}{6}$, or $\frac{2}{3} = \frac{5}{6}$, which is abſurd, if $x$ and $y$ are aſſignable Quantities, but if they are greater than any aſſignable Quantities, it may be ſaid without abſurdity, that $x = \frac{2}{3}y + \frac{2}{3}$, at the ſame Time that $x = \frac{2}{3}y + \frac{5}{6}$; becauſe the finite Magnitudes $\frac{2}{3}$ and $\frac{5}{6}$ vaniſhing in Reſpect of the infinite Magnitudes $x$ and $\frac{2}{3}y$, the two Equations $x = \frac{2}{3}y + \frac{2}{3}$ and $x = \frac{2}{3}y + \frac{5}{6}$ are reduced to $x = \frac{2}{3}y$, which involves no Contradiction.

XC.

The greateſt Difficulty which commonly occurs in the Solution of a Problem conſiſts in finding the Equations ariſing from the Conditions, becauſe it often happens that the Relations neceſſary for forming Equations are not expreſſed in the Conditions of the Problem. In this Caſe the Analyſts examine, 1°. whether ſome Quantity known or unknown, whoſe Relation with the other Quantities can be expreſſed by an Equation, may not be introduced into the Problem; and if no ſuch Quantity can be found, or if this Quantity is not ſufficient, they examine, 2°. whether among the known and unknown Quantities there be any which may be expreſſed by new Letters, whereby a new Equation may reſult. To underſtand which, let the following Problem be propoſed.

Rules for bringing problems to equations.

Problem. *There are three Meadows* a, b, c, *of the same Quality, each of a given Extent, in which the Grass grows uniformly; a Number of Oxen* d *will eat up the Pasture* a *in a Number of Days* e, *and a Number of Oxen* f *will eat up the Pasture* b *in a Number of Days* g. *It is required to find the Number of Oxen* x *which will eat up the Pasture* c *in a Number of Days* h.

It is plain that the Conditions of the Problem do not express the Relations requisite for forming Equations, but as it is Question of the Quantity of Grass in each Meadow when the Oxen entered & what grew during their Stay, I divide the Oxen of each Meadow into two Herds, I suppose the first to eat up the Grass grew in each Meadow when they entered, and the second to eat up the Grass which grows, hence I suppose $d = y + z$, $f = t + u$, $x = s + r$.

I first consider the Oxen which eat up the Grass already grew, and observing that the Number of Oxen to eat up a Meadow should be greater in Proportion as the Meadow is greater and the Time is less, I make the two Proportions $y : t = \frac{a}{e} : \frac{b}{g}$ and $y : s = \frac{a}{e} : \frac{c}{b}$, from whence I deduce $t = \frac{eby}{ag}$ and $s = \frac{ecy}{ab}$.

I next consider the Oxen which eat up the Grass which grows whilst the others eat up the Grass already grew, and observing that their Number should be greater in Proportion as the Meadows are greater, without any Regard being had to the Time. I make the two Proportions $z : u = a : b$ and $z : r = a : c$, from whence I deduce the Values of $z$ and $r$, Viz. $z = \frac{au}{b}$ and $r = \frac{cz}{a}$.

Now having seven Equations and seven unknown Quantities, $d = y + z$, $f = t + u$, $x = s + r$, $t = \frac{eby}{ag}$, $s = \frac{ecy}{ab}$, $z = \frac{au}{b}$, $r = \frac{cz}{a}$. Substituting in the third Equation for $s$ and $r$ their Values, there results $x = \frac{ecy}{ab} + \frac{cu}{b}$, and deducing from the Equations $d = y + \frac{au}{b}$, and $f = \frac{eby}{ag} + u$, $y = \frac{bdg - afg}{gb - be}$ and $u = \frac{afg - ebd}{ag - ae}$, and substituting those Values of $y$ and $u$ in the Equation $x = \frac{ecy}{ab} + \frac{cu}{b}$, I find $x = \frac{acfgb - acefg - bcdeb + bcdeg}{abgb - abeb}$.

To apply this general Solution to an Example, let the first Meadow contain $3\frac{1}{3}$ Acres, the second 10 Acres, and the third 24 Acres; and let 12 Oxen eat up the Pasture of the first in 4 Weeks, 21 Oxen eat up the Pasture of the second in 9 Weeks; and let it be required to find how many Oxen will eat up the third in 18 Weeks. By substituting those Values in the general Solution, the Number will be found to be 36.

## CHAP. II.

### *Of the Reſolution of Equations of the ſecond Degree.*

HAVING fully explained what concerns the Solution of Problems of the firſt Degree, Order requires that we ſhould paſs to thoſe of the ſecond Degree, which we propoſe to treat of in this Chapter. As to the Manner of expreſſing their Conditions, it is the ſame as for Problems of the firſt Degree, it is only to ſolve the Equations to which the Problems are brought, that different Methods are employed according to the Degrees of thoſe Equations. Of this we have an Inſtance in the following Problem, which in its full Extent includes Problems of every Degree, and is not more difficult to be expreſſed analytically in the moſt complicated Caſe as in the moſt ſimple.

I.

Problem which in its full extent includes problems of every degree.

*A Merchant having placed a Sum* a *in Trade, finding himſelf to be a looſer, is willing to withdraw at the End of the firſt Year, but having miſſed the Opportunity and not being able to obtain it until the End of the ſecond, third, or in general until the End of the* n*th Year, he finds that the Sum is diminiſhed by the Quantity* b *more than it was at the End of the firſt Year. It is required to determine how much per Cent. his Loſs amounted to yearly.*

Let $x$ be the Number ſought, that is, what each £100 loſt at the End of the firſt Year. Making the Proportion $100 : 100 - x = a : a \times \frac{100 - x}{100}$, the fourth Term $a \times \frac{100 - x}{100}$, or $a \times \left(1 - \frac{x}{100}\right)$ will expreſs what the Sum $a$ is reduced to at the End of the firſt Year.

If this Proportion be continued by ſaying $100 : 100 - x = \frac{a \times (100 - x)}{100} : \frac{a \times (100 - x)^2}{10000}$, the fourth Term $\frac{a \times (100 - x)^2}{10000}$, or $a \times \left(1 - \frac{x}{100}\right)^2$ will expreſs what the Sum $a$ is reduced to at the End of the ſecond Year, and what the ſame Sum $a$ is reduced to at the End of the third Year will be expreſſed by $a \times \left(1 - \frac{x}{100}\right)^3$, and in general what it is reduced to at the End of the $n$th Year will be expreſſed by $a \times \left(1 - \frac{x}{100}\right)^n$, that is, by $a$ multiplied by the Quantity $1 - \frac{x}{100}$ raiſed to the Power $n$.

II.

If now it was propoſed to find the Equation to be ſolved, ſuppoſing the Merchant to have withdrawn at the End of the ſecond Year, it is

Equation of the foregoing problem for the second degree

manifeſt that the Quantity $a \times \left(1 - \frac{x}{100}\right)^2$ ſhould be put equal to the Quantity $a \times \left(1 - \frac{x}{100}\right)$ diminiſhed by the Quantity $b$, which will give $a \times \left(1 - \frac{x}{100}\right)^2 = a \times \left(1 - \frac{x}{100}\right) - b$, or multiplying $1 - \frac{x}{100}$ by itſelf, as indicated by the Exponent 2,

$a \times \left(1 - \frac{2x}{100} + \frac{x^2}{10000}\right) = a \times \left(1 - \frac{x}{100}\right) - b$, which is reduced to $x^2 - 100x = -10000 \frac{b}{a}$, Equation of the ſecond Degree, for to ſolve which the foregoing Methods are inſufficient.

III.

For the third degree

If the Merchant is ſuppoſed to withdraw at the End of the third Year, the Equation to be ſolved will be $a \times \left(1 - \frac{x}{100}\right)^3 = a \times \left(1 - \frac{x}{100}\right) - b$, which by multiplying $1 - \frac{x}{100}$ twice by itſelf, as the Exponent 3 indicates, becomes

$a \times \left(1 - \frac{3x}{100} + \frac{3x^2}{10000} - \frac{x^3}{1000000}\right) = a \times \left(1 - \frac{x}{100}\right) - b$,

or $x^3 - 300x^2 + 20000x = 1000000 \frac{b}{a}$, an Equation more difficult to be ſolved than the foregoing.

IV.

As to the other Caſes it is eaſy to perceive how the Equations they furniſh may be formed, and that the Equation will be always of a Degree expreſſed by the Number $n$. If this Equation in general without ſpecifying the Number $n$ be required, it may be obtained by employing the general Expreſſion $a \times \left(1 - \frac{x}{100}\right)^n$ of the Quantity to which $a$ is reduced to after the $n$th Year, and the Equation will be

$a \times \left(1 - \frac{x}{100}\right)^n = a \times \left(1 - \frac{x}{100}\right) - b$, or

$\left(1 - \frac{x}{100}\right)^n = 1 - \frac{x}{100} - \frac{b}{a}$.

V.

We ſhall for the preſent confine ourſelves to the Inveſtigation of the Solution of that Caſe of the Problem in which its Equation is of the ſecond Degree, that is, when it is $x^2 - 100x = -10000 \frac{b}{a}$, or ra-

ther we ſhall inveſtigate a Method for ſolving in general all Equations of the ſecond Degree. Thoſe who are deſirous of ſolving the higher Caſes of the ſame Problem will eaſily effect it after they have ſeen the general Methods to be explained hereafter correſponding to the different Degrees of thoſe Equations.

Inveſtigation of the method for ſolving equations of the ſecond degree

What moſt naturally occurs in ſearching for a Method for ſolving in general Equations of the ſecond Degree, is to examine the Relation which ſubſiſts between thoſe Equations and the Equations of the firſt Degree. Now it is evident that every Equation of the firſt Degree will become one of the ſecond, if the two Members be ſquared, for Example, $x + a = b$ becomes, when ſquared, $x^2 + 2ax + a^2 = b^2$, it remains therefore to know whether by a contrary Operation every Equation of the ſecond Degree may not be reduced to one of the firſt.

Let us take, for Example, the Equation $x^2 - px = -q$, and let us try if $x^2 - px = -q$ is not the Square of ſome Quantity, the firſt Part of which is $x$ and the ſecond a known Quantity, in order to find by this Means the Equation of the firſt Degree, which being ſquared, would become $x^2 - px = -q$. Now it is eaſy to perceive that $x^2 - px$ is not a Square, but at the ſame Time it is manifeſt that it may be made one by an Addition of ſome Quantity, and we are at Liberty to make this Addition, provided the ſame Quantity be added to the other Side of the Equation.

To find this Quantity which added to $x^2 - px$ will render it a compleat Square, we have no more to do than to compare it with the Square $x^2 + 2ax + a^2$, the Term $px$ correſponding with $2ax$, $p$ will correſpond to $2a$, and conſequently $a$ to $\frac{1}{2}p$. Now as $a^2$ is what compleats $x^2 + 2ax$ into a Square, the Square of $-\frac{1}{2}p$, that is, $\frac{1}{4}p^2$ will compleat $x^2 - px$ into a Square, that is, $x^2 - px + \frac{1}{4}p^2$ will be a Square, it is one in effect, Viz. that of $x - \frac{1}{2}p$ when $x > \frac{1}{2}p$, and that of $\frac{1}{2}p - x$ when $x < \frac{1}{2}p$. Having therefore added $\frac{1}{4}p^2$ to the firſt Member of the Equation, the ſame muſt be added to the ſecond, and there reſults $x^2 - px + \frac{1}{4}p^2 = \frac{1}{4}p^2 - q$. Now either the Quantity $x - \frac{1}{2}p$ or the Quantity $\frac{1}{2}p - x$ multiplied by itſelf, gives $x^2 - px + \frac{1}{4}p^2$, wherefore $x - \frac{1}{2}p$ or $\frac{1}{2}p - x$ will be equal to the Number which multiplied by itſelf will give $\frac{1}{4}p^2 - q$, which is expreſſed in general thus, $\surd(\frac{1}{4}p^2 - q)$, the Sign $\surd$ being employed by the Analyſts to indicate that the ſquare Root of the Quantity before which it is placed is to be extracted.

The ſign $\surd$ indicates the ſquare root.

Employing therefore this Denomination $x - \frac{1}{2}p = \surd(\frac{1}{4}p^2 - q)$ and $\frac{1}{2}p - x = \surd(\frac{1}{4}p^2 - q)$, from whence is deduced $x = \frac{1}{2}p \pm \surd(\frac{1}{4}p^2 - q)$. The Sign $\pm$ being employed by the Analyſts to include in one and the ſame Expreſſion the two Values of $x$.

An equation of the ſecond degree has two roots.

VI.

Let us now apply this general Solution to the Equation $x^2 - 100\,x = -10000\,\frac{b}{a}$, to which we were led in the foregoing Problem. Comparing this Equation with $x^2 - px = -q$, we will have $p = 100$, $q = 10000\,\frac{b}{a}$, and ſubſtituting thoſe Values in the Place of $p$ and $q$ in the general Formula $x = \frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 - q}$, we will have $x = 50 + \sqrt{\left(2500 - 10000\,\frac{b}{a}\right)}$.

VII.

Reduction of the value of $x$ by reſolving the root of the product into thoſe of its factors.

We may reduce to a more ſimple Form the radical Part $\sqrt{\left(2500 - 10000\,\frac{b}{a}\right)}$ of this Value of $x$, by obſerving, that the ſquare Root of the Product of two or more Quantities is the Product of the ſquare Roots of thoſe Quantities; for reſolving $2500 - 10000\,\frac{b}{a}$ into its Factors $2500$, $1 - \frac{4b}{a}$, and extracting the Roots of thoſe two Quantities we will have $50$ and $\sqrt{\left(1 - \frac{4b}{a}\right)}$ whoſe Product $50\sqrt{\left(1 - \frac{4b}{a}\right)}$ will be $\sqrt{\left(2500 - 10000\,\frac{b}{a}\right)}$ that is, that the Value of $x$ will be $50 \pm 50\sqrt{\left(1 - \frac{4b}{a}\right)}$.

As to the Demonſtration of this Principle, that the Root of any Product is obtained by multiplying the Roots of its Factors, it eaſily may be found, by obſerving that the Square of a Product, as $a\,b$, is the Product $a\,a \times b\,b$, of the Squares $a\,a$ & $b\,b$ of its Factors $a$ and $b$.

VIII.

Example of this problem.

To make Uſe of this Value of $x$, nothing more is required than to know the Ratio that $b$ bears to $a$. Let $b$, for Example, be the $\frac{6}{25}$ of $a$, that is, let us ſuppoſe the Merchant to have found the Sum at the End of the ſecond Year, diminiſhed by a Quantity equal to the $\frac{6}{25}$ of the Whole more than it was at the End of the firſt Year, according to this Suppoſition $\frac{4b}{a} = \frac{24}{25}$ and $1 - \frac{4b}{a} = \frac{1}{25}$, hence the Root $\sqrt{\left(1 - \frac{4b}{a}\right)} = \frac{1}{5}$ and conſequently will give $x = 50 \pm 50 \times \frac{1}{5} = 60$ or $40$.

In effect thoſe two Values of $x$ reſolve the Equation $x^2 - 100\,x = -2400$ into which the Equation $x^2 - 100\,x = -10000\,\frac{b}{a}$ is transformed by the Suppoſition of $\frac{b}{a} = \frac{6}{25}$, for whether $x$ be ſuppoſed $= 60$ or $40$, $x^2 - 100\,x$ becomes $= -2400$.

The Neceſſity of the two Solutions 60 and 40 will appear after a more ſatisfactory Manner, as follows. Let us ſuppoſe firſt $x = 60$, that is, that the Sum of £100000, for Example, is diminiſhed 60 per Cent. yearly, it is manifeſt that at the End of the firſt Year it would be reduced to £40000, and at the End of the ſecond Year it will be reduced to £16000; but £16000 are leſs than £40000 by £24000, which are the $\frac{6}{25}$ of £100000, as the Problem requires.

Let us now ſuppoſe the ſame Sum of £100000 to be diminiſhed 40 per Cent. yearly, at the End of the firſt Year it will be reduced to £60000, and at the End of the ſecond to £36000, but £36000 are leſs than the Sum £60000 by the Quantity £24000, or the $\frac{6}{25}$ of £100000.

IX.

If $b$ was ſuppoſed to be the $\frac{4}{25}$ of $a$, then $x = 50 \pm 50\sqrt{(1 - \frac{16}{25})} = 50 \pm 50\sqrt{\frac{9}{25}} = 50 \pm 30$, that is, 80 or 20, both of which will be found to ſolve the Problem. Another example.

X.

If $\frac{b}{a}$ be put $= \frac{1}{3}$, there will reſult $x = 50 \pm 50\sqrt{(1 - \frac{4}{3})}$, or $50 \pm 50\sqrt{-\frac{1}{3}}$, but as the Squares of all Quantities are affirmative, it follows that the Quantity $\sqrt{-\frac{1}{3}}$ cannot be aſſigned, or which comes to the ſame that the Problem is impoſſible in this Caſe. Third example in which the root of a negative quantity being required is impoſſible

Hence we may conclude that there is no poſſible Value which ſubſtituted for $x$ in the Equation $x^2 - 100x = -\frac{10000}{3}$ will render the two Members equal, or what comes to the ſame Thing, that the Sum $a$ cannot be leſſened each Year according to any given Proportion, ſo that from the ſecond to the third Year the Diminution may amount to a Quantity equal to a Third of the whole Sum. The Analyſts however regard as a Kind of a Solution or Root of the Equation $x^2 - 100x = -\frac{10000}{3}$, the Value $50 \pm 50\sqrt{-\frac{1}{3}}$ which reſults from thence, but they call it an imaginary Root, and this imaginary Root on Account of the Sign $\pm$ is always conſidered as a double Solution. Thoſe roots are called imaginary.

From the general Value of $x$, $\frac{1}{2}p \pm \sqrt{(\frac{1}{4}p^2 - q)}$, it appears that as often as the Quantity deſigned by $q$ is negative and greater than $\frac{1}{4}p^2$, the two Roots of the Equation $x^2 - px = -q$ will be both imaginary. Equations of the ſecond degree whoſe roots are imaginary.

XI.

Though the Roots of an Equation or the Values of the unknown Quantity are real and poſitive, however they do not always, as in the foregoing Example both ſolve the Problem. Let the following Queſtion be propoſed. The real and poſitive values of the unknown quantity do not always ſolve the problem

*The Sum of* £190 *was divided between three Perſons; whoſe Shares were in geometrical Proportion, and the greateſt of them exceeded the leaſt by* £50. *What were the ſeveral Shares?*

If $x$ be put for the leaſt of them, then the greateſt will be $x + 50$, the Sum of which two ſubtracted from (190) the Whole, leaves $140 - 2x$ First example.

for the mean Share. Therefore $x : 140 - 2x = 140 - 2x : x + 50$, and consequently $(140 - 2x)^2 = x(x + 50)$, that is, $19600 - 560x + 4x^2 = x^2 + 50x$, whence $3x^2 - 610x = -19600$, and $x^2 - \frac{610x}{3} = -\frac{19600}{3}$, hence by compleating the Square $x^2 - \frac{610x}{3} + \frac{93025}{9} = \frac{19600}{3} + \frac{93025}{9} = \frac{34225}{9}$, and by extracting the Root $x = \frac{305}{3} \pm \frac{185}{3}$, or $x = 40$ and $x = 163\frac{1}{3}$, two real and positive Roots, however the first Root only solves the Problem, the other two Shares being 60 and 90 Pounds.

As to the Root $163\frac{1}{3}$ it solves the following Question.

*A Sum of Money was divided between three Persons, whose Shares were in geometrical Proportion, the greatest exceeding the least by £50, and their Sum exceeding the mean Share by £190. What were the several Shares?*

For the analytical Expression of this Question is $(2x - 140)^2 = x^2 + 50x$, from whence results the same Equation as in the foregoing Question, and in this Case $x = 163\frac{1}{3}$. Because the Equation $19600 - 560x + 4x^2 = x^2 + 50x$, includes not only $(140 - 2x)^2 = x^2 + 50x$, but also $(2x - 140)^2 = x^2 + 50x$, which are the analitick Translations of two very different Questions, the first of which gives $x = 40$, the latter $163\frac{1}{3}$, wherefore the Roots of an Equation, though real and positive, do not both solve the Question in the Sense it is proposed, but considered in two different Lights, which Difference cannot be expressed analitically. For a further Illustration, here follows another Example.

XII.

Second example.

*Two Travellers set out at the same Time, the one from* Dublin, *the other from* Londonderry, *the former for* Londonderry, *the latter for* Dublin; *when they met and had computed their Journey, it was found, that the former had travelled thirty Miles more than the latter, and that at their Rate of Travelling, the former expected to reach* Londonderry *in four Days, and the latter to reach* Dublin *in nine Days; the Distance between* Dublin *and* Londonderry *is required.*

I put $x$ for the Number of Miles between *Dublin* and *Londonderry*; and since the Travellers both together had travelled $x$ Miles when they met; it follows that the former had travelled $\frac{x + 30}{2}$ Miles, and the latter $\frac{x - 30}{2}$ Miles, according to the first Condition of the Problem, therefore the remaining Part of the formers Journey to *Londonderry* is $\frac{x - 30}{2}$ Miles, which he expects to perform in four Days, and the

remaining Part of the latters Journey is $\frac{x+30}{2}$ Miles, which he expects to perform in nine Days. I now enquire into the Number of Days each hath travelled already, and I say $\frac{x-30}{2} : \frac{x+30}{2} = 4 : \frac{4\times(x+30)}{x-30}$ Number of Days travelled by the Former, and $\frac{x+30}{2} : \frac{x-30}{2} = 9 : \frac{9\times(x-30)}{x+30}$ Number of Days travelled by the Latter, but as they both set out at the same Time and are now met, they must have travelled the same Number of Days, therefore $\frac{4\times(x+30)}{x-30} = \frac{9\times(x-30)}{x+30}$, which after the necessary Reductions, becomes $x^2 - 156x = -900$, whence by compleating the Square $x^2 - 156x + 6084 = 5184$, consequently $x = 150$ or $x = 6$, therefore the Distance between *Dublin* and *Londonderry* must either be 150 or 6 Miles; but 6 Miles it cannot be, because when the first Traveller came up to the Latter, he had travelled 30 Miles more than him, and had not yet reached *Londonderry*; therefore the Distance between *Dublin* and *Londonderry* is 150 Miles; for then the first Traveller must have made 75 + 15 or 90 Miles, and the second 75 — 15 or 60 Miles, from the Time of their setting out; therefore the Former has 60 Miles and the Latter 90 Miles to travel; but if the Former could travel 60 Miles in 4 Days, he must at the same Rate have travelled 90 Miles in 6 Days, and if the Latter could travel 90 Miles in 9 Days, he must have travelled 60 Miles in 6 Days, as the Problem requires.

As to the second Value, $x = 6$, it solves the following Problem.

*Two travellers set out at the same Time, the one from* Dublin, *the other from* Swords, *the Former with a Design to pass through* Swords, *and the Latter with a Design to travel the same Way. The Former had overtaken the Latter, and having computed their Journey, it was found that they had both together travelled* 30 *Miles, that the Former had passed through* Swords 4 *Days before, and that the Latter at his Rate of Travelling, was a nine Days Journey distant from* Dublin. *The Distance of* Dublin *from* Swords *is required.*

For if $x$ be put for the Number of Miles from *Dublin* to *Swords*, then it is plain that the first Traveller must have travelled more Miles than the Latter by $x$; but they both together travelled 30 Miles when they met, wherefore the first must have travelled $\frac{30+x}{2}$, and the Latter must have travelled $\frac{30-x}{2}$ Miles, therefore the Distance of the first from *Swords*, after he had overtaken the second, was $\frac{30-x}{2}$ Miles, which

he had travelled in 4 Days, and the Diſtance of the ſecond from *Dublin* was $\frac{30+x}{3}$ Miles, which he could travel in 9 Days; therefore to find how many Days each had trrvelled already, I ſay $\frac{30-x}{2} : \frac{30+x}{3} = 4 : \frac{4\times(30+x)}{30-x}$ Number of Days travelled by the Former, and $\frac{30+x}{2} : \frac{30-x}{2} = 9 : \frac{9\times(30-x)}{30+x}$ Number of Days travelled by the Latter, and as they both ſet out at the ſame Time, they muſt have both travelled the ſame Number of Days, wherefore $\frac{4\times(30+x)}{30-x} = \frac{9\times(30-x)}{30+x}$, which after the neceſſary Reductions, becomes $x^2 - 156x = -900$, the ſame Equation as found above, whoſe Roots are $x = 150$ and $x = 6$. Wherefore the Diſtance between *Dublin* and *Swords* muſt either be 6 or 150 Miles, but 150 Miles it cannot be, becauſe after the firſt Traveller had paſſed from *Dublin* beyond *Swords*, and at laſt had overtaken the ſecond, they had both travelled but 30 Miles, therefore the Diſtance from *Dublin* to *Swords* muſt be 6 Miles, and this Number will anſwer the Conditions of the Problem, for then the firſt Traveller when he had overtaken the Latter, had travelled 15 + 3 or 18 Miles, and the Latter 15 — 3 or 12 Miles. Therefore the firſt Traveller had got 12 Miles beyond *Swords* in 4 Days Time, and the Latter was 18 Miles diſtant from *Dublin*, which he could travel in 9 Days, but at the Rate of 12 Miles in 4 Days, the Former muſt have performed his 18 Miles Journey in 6 Days, and at the Rate of 18 Miles in 9 Days, the Latter muſt have performed his 12 Miles Journey alſo in 6 Days, therefore from the Time of their firſt ſetting out to the Time of their Meeting, they had both travelled the ſame Number of Days as the Problem requires.

XIII.

General rule for reſolving equations of the ſecond degree.

It is eaſy to perceive that any Equation of the ſecond Degree may be reſolved by repeating the Proceſs purſued in reſolving the foregoing Equations, which may be expreſſed thus. 1°. Tranſpoſe all the Terms that involve the unknown Quantity to one Side, and the known Terms to the other Side of the Equation. 2°. If the Square of the unknown Quantity is affected with a Coeficient, you are to divide all the Terms by that Coeficient, and if the Sign of that Power be negative you are to change all the Signs of the Terms. 3°. Add to both Sides the Square of half the Coeficient prefixed to the unknown Quantity itſelf, and the Side of the Equation that involves the unknown Quantity will then be a compleat Square. 4°. Extract the Square Root from both Sides of the Equation, which you will find on one Side always to be the unknown

Quantity, with half the foreſaid Coeficient ſubjoined to it, ſo that by tranſpoſing this half you may obtain the Value of the unknown Quantity expreſſed in known Terms, for Example, ſuppoſe $x^2 - 6x = 27$, adding 9 the Square of the half of 6 to both Sides, there reſults $x^2 - 6x + 9 = 9 + 27 = 36$, and extracting the Square Root you will find $x - 3 = 6$ and $3 - x = 6$, according as $x$ is $>$ or $< 6$, wherefore $x = +3 \pm 6$ or $x = 9$ and $x = -3$, which both reſolve the Equation $x^2 - 6x = 27$, in like Manner if $x^2 + 8x = 9$, adding 16 Square of the half of 8 to both Sides, there will reſult $x^2 + 8x + 16 = 16 + 9 = 25$, and extracting the Square Root you will find $x + 4 = \pm 5$, that is, $x = -4 \pm 5$, or $x = -9$ and $x = 1$.

XIV.

To accuſtom Beginners to the Difficulties which occur in the Solution of Problems of the ſecond Degree, here follows another Problem.

Another problem of the ſecond degree.

*Let it be propoſed to find in the Line which joins two Lights the Point where a Body would be equally illuminated by them, according to this Principle of Phyſicks, that the Effect of a Light is four Times greater when it is twice nearer, nine Times greater when it is three Times nearer, or increaſes as the Diſtance decreaſes.*

Let $a$ expreſs the Diſtance between the two given Lights, and let the Ratio of $m$ to $n$ expreſs that of the Effect of the leaſt Light at a certain Diſtance, to the Effect of the greateſt Light at the ſame Diſtance.

Let $x$ expreſs the Diſtance of the leaſt of the two Lights from a Point taken at will in the Line which joins the two Lights, it is manifeſt that $a - x$ will expreſs the Diſtance of the other Light from the ſame Point, that the Squares of thoſe two Diſtances will be $x^2$ and $x^2 - 2ax + a^2$, and conſequently the Quantities which decreaſe as the Squares of thoſe Quantities increaſe will be to one another as $\frac{1}{x^2}$ and $\frac{1}{x^2 - 2ax + a^2}$.

From whence it follows that if the Lights were of equal Force, the Effects they would each produce in this ſame Point would be to each other as $\frac{1}{x^2}$ to $\frac{1}{x^2 - 2ax + a^2}$; but the abſolute Quantities of thoſe Lights being to each other in the Ratio of $m$ to $n$, their Effects therefore will be to each other as $\frac{m}{x^2}$ to $\frac{n}{x^2 - 2ax + a^2}$.

Now that the Point taken at will may become the Point required, I put thoſe two Quantities equal to each other, which gives the Equation $ma^2 - 2amx + mx^2 = nx^2$.

To ſolve this Equation, I tranſpoſe the Terms $mx^2$ and $2amx$ into the other Member, and there reſults $(n - m)xx + 2amx = maa$.

or $xx + \frac{2am}{n-m}x = \frac{aam}{n-m}$. I afterwards add to the two Members of this Equation the Square of half the Coeficient of the ſecond Term, and there reſults $x^2 + \frac{2amx}{n-m} + \frac{a^2m^2}{(n-m)^2} = \frac{aam}{n-m} + \frac{aamm}{(n-m)^2}$, whoſe ſecond Member becomes $\frac{a^2mn}{(n-m)^2}$, by reducing the two Terms $\frac{aam}{n-m} + \frac{aamm}{(n-m)^2}$ to the ſame Denominator.

Then extracting the Square Root of the two Members, there reſults $x + \frac{am}{n-m} = \pm\sqrt{\frac{aamn}{(n-m)^2}}$, or $x = -\frac{am}{n-m} \pm \frac{a}{n-m}\sqrt{mn}$.

By extracting the Root of the Part $\frac{aa}{(n-m)^2}$ which is a perfect Square, and leaving under the radical Sign its Multiplier $mn$, which is no Square, at leaſt for all the Values of $m$ and $n$. Wherefore the two Values of $x$ which ſolve the foregoing Equation, and conſequently the Problem which leads to this Equation are expreſſed by the Formula $x = -\frac{am}{n-m} \pm \frac{a}{n-m}\sqrt{mn}$, or $x = \frac{a}{n-m} \times -m \pm \sqrt{mn}$.

XV.

Of the two foregoing values, one is neceſſarily poſitive and the other negative.

From this Expreſſion it appears that one of the Values is neceſſarily negative and the other poſitive, for 1°. if we take the radical Quantity $\sqrt{mn}$ with the Sign —, there is no doubt but the whole Quantity will be negative, 2°. if we take $\sqrt{mn}$ with the Sign +, $-m + \sqrt{mn}$ which we will have then will be poſitive, becauſe $n$ being greater $m$, $\sqrt{mn}$ muſt be greater than $m$.

XVI.

Uſe of the negative value.

If we now enquire into the Uſe of the negative Value of the Diſtance $x$ of the leaſt Light to the aſſumed Point, we will find by recalling what had been demonſtrated (Chap. I. Art. LXIII.) with Reſpect to thoſe Values in Equations of the firſt Degree, that its Direction is oppoſite to that of the firſt, that is, that the Point which it gives for ſolving this Problem, inſtead of being placed between the two Lights, will be placed in the Line on the other Side of the weakeſt Light.

There can be no Difficulty in admitting this Poſition of the negative Value of $x$, when it is obſerved, that this ſame Value was found negative only, becauſe the Problem was ſolved in the Suppoſition that the Point ſought was placed between the two Lights; for if it had been ſuppoſed (as it might be) to be placed in the Line which joins them, produced on the Side of the weakeſt Light, there would have reſulted another Computation relative to this Poſition, and $x$ which in this Caſe

is taken in the Line produced on the Side of the weakeſt Light would be poſitive.

XVII.

In order to make this appear, we will reſume the Problem, the Point ſought being ſuppoſed in the Line produced on the Side of the weakeſt Light. The Diſtance of this Point from the leaſt Light being expreſſed by $x$, its Diſtance from the greateſt Light will be expreſſed by $a + x$, the Squares of thoſe Diſtances will be $x^2$ and $a^2 + 2ax + x^2$, and the two Quantities of Light $\frac{m}{x^2}$ and $\frac{n}{a^2 + 2ax + x^2}$, which being equal by the Conditions of the Problem, will give

$\frac{m}{x^2} = \frac{n}{a^2 + 2ax + x^2}$, or $ma^2 + 2amx + mx^2 = nx^2$,

or $(n - m)x^2 - 2amx = ma^2$, or $x^2 - \frac{2amx}{n - m} = \frac{m}{n - m}a^2$,

which being reſolved will give $x = \frac{a \times (m \pm \sqrt{mn})}{n - m}$, the firſt Value of which $\frac{a \times (m + \sqrt{mn})}{n - m}$ will be poſitive, and ſolves the Problem in the Senſe it is propoſed.

As to the ſecond Value $\frac{a \times (m - \sqrt{mn})}{n - m}$ of the Diſtance $x$, being negative, it ſhould be taken in an oppoſite Direction to the firſt, that is, the Point which it gives is not ſituated in the Line, which joins the two Lights produced, but in the Line itſelf.

Hence in this Solution the Values of $x$ differ from thoſe in the former with Reſpect to the Signs, and thoſe two Solutions confirm what has been already proved in the foregoing Chapter, Art. LXIII. that the unknown Quantities which become negative, ſhould be always conſidered as being of an oppoſite Kind from what they have been ſuppoſed in expreſſing the Conditions of the Problem.

XVIII.

In order to remove whatever Difficulties may occur to the Reader with Reſpect to this Problem, we ſhall apply it to an Example. Let us ſuppoſe $n = 4m$, that is, that the greateſt Light has four Times the Force of the other. Subſtituting this Value of $n$ in the general Formula of Art. XIV. $x = \frac{a}{n - m} \times (m \pm \sqrt{mn})$, it will be transformed into $x = \frac{a}{3} \times \pm (2 - 1)$, that is, $+\frac{1}{3}a$, or $-a$, which furniſh two Points, both of which ſolve the Problem, one ſituated between

Example of the foregoing problem

the two Lights twice nearer the weakeſt than the ſtrongeſt, and the other in the line produced, at a Diſtance from the weakeſt equal to the Diſtance between the two Lights.

XIX.

The poſitive & negative values of the unknown quantity do not always both ſolve the problem

The poſitive and negative Values of the unknown Quantity do not always, as in the foregoing Example both ſolve the Problem. Let the following Queſtion be propoſed.

*A Draper buys ſome Ells of Cloth for 70 Crowns, and finds that if he had 4 Ells more, he had bought every Ell two Crowns cheaper. How many Ells did he buy?*

Let $x$ denote the Number of Ells fought, then dividing the whole Price by the Number of Ells, we have $\frac{70}{x}$ for the Price of one Ell, and $\frac{70}{x+4}$ for the Price of one Ell if he had got 4 Ells more, conſequently $\frac{70}{x} - 2 = \frac{70}{x+4}$ whence $x^2 + 4x = 140$, conſequently $x = -2 \pm 12$, or $x = 10$ and $x = -14$, the firſt Value of $x$ ſolves the Problem, as to $x = -14$, it does not ſolve the propoſed Queſtion, it ſolves the following.

*A Draper buys ſome Ells of Cloth for 70 Crowns, and finds that if he had 4 Ells leſs, he would have bought every Ell 2 Crowns dearer. How many Ells did he buy?*

In Effect the analitick Expreſſion of this Queſtion is $\frac{70}{x} + 2 = \frac{70}{x-4}$, or $x^2 - 4x = 140$, from whence is deduced $x = 14$ and $x = -10$, which are preciſely the Roots of the foregoing Equation with contrary Signs. From whence it appears that the negative Roots ſolve the Queſtion, not as it is propoſed, but with ſome Alterations which conſiſt in ſubtracting what ſhould be added, or in adding what ſhould be ſubtracted. The Sign — which precedes the Root $x = -14$, for Example, indicating that in the foregoing Queſtion the 4 Ells are to be ſubtracted, and the 2 Crowns to be added to the Price.

XX.

When a queſtion is wrong ſtated the values of the unknown quantity become both negative.

If the Roots of an Equation or Values of the unknown Quantity are both negative, then the Queſtion has been wrong ſtated, as will appear by the following Example.

A *and* B *take in Trade* £2112 *per Annum each, but* A *whoſe Profits are* 2 *per Cent. greater than thoſe of* B, *clears* £100 *per Annum more than* B. *What are the Profits of each per Cent. and what do they clear per Annum.*

Let $x$ denote what *A* gains per Cent. Now ſince *A* for every £100 he layed out in purchaſing Goods, recovered in the Sales $100 + x$ Pounds, and 2112 being the whole Sum recovered, $100 + x : x = 2112 :$

$\frac{2112\,x}{100+x}$ the whole Profit of *A*, and in like Manner $98+x : x-2 = 2112 : \frac{2112\,x-4224}{98+x}$ the whole Profit of *B*. Therefore $\frac{2112\,x}{100+x} - \frac{2112\,x-4224}{98+x} = 100$ by the Queſtion, hence $x^2 + 198\,x = -5576$, whence by compleating the Square and extracting the Root $x = -99 \pm 65$, both which Roots are negative, which denotes that the Queſtion ſhould be expreſſed thus:

A *and* B *take in Trade* £2112 *per Annum each, but* A *whoſe Loſſes are* 2 *per Cent. leſs than thoſe of* B, *loſes* 100 *per Annum leſs than* B. *What are the Loſſes each ſuſtained per Cent. and what did their Loſſes amount to yearly?*

In effect, ſince *A* for every £100 he layed out in purchaſing Goods, recovered but $100-x$ Pounds, and £2112 being the whole Sum recovered, $100-x : x = 2112 : \frac{2112\,x}{100-x}$ the whole Loſs of *A*, in like Manner $98-x : x+2 = 2112 : \frac{2112\times(x+2)}{98-x}$ the whole Loſs of *B*, wherefore $\frac{2112\,x+4224}{98-x} - \frac{2112\,x}{100-x} = 100$ by the Queſtion. Hence $x^2 - 198\,x = -5576$, and compleating the Square and extracting the ſquare Root $x = 99 \pm 65$, which are the ſame Roots as found before but affirmative.

From which it appears that *A* loſt £34 per Cent. and his whole Loſs amounted to £1088 per Annum, and that *B* loſt £36 per Cent. and his whole Loſs amounts to £1188 per Annum.

XXI.

The Principles which we have explained are ſufficient for ſolving all Equations of the ſecond Degree, but to render the Application of them eaſy to Beginners, we ſhall proceed to exerciſe them in the Reſolution of ſeveral Equations, there will reſult this Advantage, that beſides their becoming better acquainted with the Method, they will learn at the ſame Time the new Operations of ſpecious Arithmetick, which, without doubt, are owing to the Reſearches which the firſt Analyſts have made on the Equations of the ſecond Degree.

Other examples of the reſolution of equations of the ſecond degree.

Let $b\,x^2 = 2\,c^2\,x + 2\,c^2\,a$, tranſpoſing the Terms affected with $x$ on one Side, and dividing all the Terms by the Coeficient of $x\,x$, we will have $x^2 - \frac{2\,c^2\,x}{b} = \frac{2\,c^2\,a}{b}$, to the two Members of which, adding $\frac{c^4}{b^2}$, and afterwards extracting the Square Root, we will have

$$x = \frac{c^2}{b} \pm \sqrt{\left(\frac{2\,c^2\,a\,b + c^4}{b^2}\right)} = \frac{c^2 \pm c\sqrt{(2\,a\,b + c^2)}}{b}.$$

Let $f^2 + g^2 - 2gx + x^2 = \frac{m^2 x^2}{n^2}$ which I first reduce to $\left(\frac{mm - nn}{nn}\right) \times x^2 + 2gx = f^2 + g^2$, and afterwards to $x^2 + \frac{2gn^2}{m^2 - n^2} x = \frac{f^2 n^2 + g^2 n^2}{m^2 - n^2}$, or $x^2 + \frac{2gn^2 x}{m^2 - n^2} + \frac{g^2 n^4}{(m^2 - n^2)^2} = \frac{f^2 n^2 + g^2 n^2}{m^2 - n^2} + \frac{g^2 n^4}{(m^2 - n^2)^2} = \frac{ffnnmm + ggnnmm - ffn^4}{(mm - nn)^2}$, from whence is deduced $x + \frac{gnn}{mm - nn} = \pm \frac{n\sqrt{(ffmm + ggmm - ffnn)}}{mm - nn}$, or $x = \frac{n}{mm - nn} \times [-gn \pm \sqrt{(ffmm + ggmm - ffnn)}]$.

Let $abc - af^2 + 2afz = az^2 - bz^2$ be given, whose Terms being disposed in order, will become $z^2 - \frac{2af}{a-b} z = \frac{abc - af^2}{a-b}$, and compleating the Square $z^2 - \frac{2af}{a-b} z + \frac{a^2 f^2}{(a-b)^2} = \frac{aabc - abbc + abff}{(a-b)^2}$, hence $z = \frac{af \pm \sqrt{(aabc - abbc + abf^2)}}{a-b}$.

Let the Equation $4a^2 - 2x^2 + 2ax = 18ab - 18b^2$ be proposed, disposing its Terms in order and reducing it, there results $x^2 - ax = 2a^2 - 9ab + 9b^2$, and compleating the Square, becomes $x^2 - ax + \frac{1}{4}a^2 = \frac{9}{4}a^2 - 9ab + b^2$, which gives $x = \frac{1}{2}a \pm \sqrt{(\frac{9}{4}a^2 - 9ab + 9b^2)}$.

The young Analyst would have easily reduced this Quantity if he had perceived, and he could not but have perceived by what precedes, that the Square of a Quantity composed of two Terms, is equal to the Sum of the Squares of each of those two Terms, and to double of the Product of those two Terms.

For finding in the Quantity $\frac{9}{4}a^2 - 9ab + 9b^2$, the Terms $\frac{9}{4}a^2$ and $9b^2$, which are the Squares of $\frac{3}{2}a$ and $3b$, and the Term $9ab$ which is double the Product of $\frac{3}{2}a$ into $3b$. It is easy to conclude that this Quantity $\frac{9}{4}a^2 - 9ab + 9b^2$ is the Square of $\frac{3}{2}a - 3b$, therefore instead of the Expression $\sqrt{(\frac{9}{4}a^2 - 9ab + 9b^2)}$, we may write simply $\frac{3}{2}a - 3b$: Wherefore the Value of $x$ is $\frac{1}{2}a \pm \frac{3}{2}a \mp 3b$, that is, either $2a - 3b$, or $-a + 3b$. In effect, both those Values solve the given Equation.

XXII.

Process of the extraction of the square root

Among the different Equations of the second Degree, which are required to be resolved, Cases similar to the foregoing may occur; it is therefore necessary to have a general Method for discovering such Quan-

tities as are Squares, and for finding their Roots; this Method is easily deduced from the Principles employed in the foregoing Example. The Proceſs of this Method applyed to an Exampſe, is as follows. explained in an example.

Let the ſquare Root of the Quantity $30\,a\,b + 9\,b^2 + 25\,a^2$ be required.

$$\begin{array}{r|l} 25\,a^2 + 30\,b\,a + 9\,b^2 & 10\,a + 3\,b \\ -\,25\,a^2 & 5\,a + 3\,b \\ \hline 30\,b\,a + 9\,b^2 & \\ -\,30\,b\,a - 9\,b^2 & \\ \hline 0 & \end{array}$$

I firſt range the Terms of this Quantity according to the Dimenſions of the Letter $a$, for Example, as above.

I afterwards extract the Root of the firſt Term $25\,a^2$ which is $5\,a$, which will be the firſt Term of the Root, and I write it down beſide the propoſed Quantity $25\,a^2 + 30\,b\,a + 9\,b^2$, ſeparating them by a Line in order to avoid Confuſion. I then write under the propoſed Quantity the Square $25\,a^2$ of $5\,a$, prefixing to it the Sign —. I draw a Line and reduce, and there remains $30\,a\,b + 9\,b^2$, which I write under the Line, which being done, I double $5\,a$ which gives $10\,a$, and divide the firſt Term $30\,a\,b$ of the Quantity $30\,a\,b + 9\,b^2$ by $10\,a$, and I write the Quotient $3\,b$, which is the ſecond Term of the Root ſought, beſide $5\,a$, and I place it at the ſame Time beſide $10\,a$, and I multiply the ſuperior Quantity by this new Term $3\,b$ of the Root, obſerving as in Diviſion to change the Signs, in writing the Product under the Quantity $30\,a\,b + 9\,b^2$, then reducing and finding that all the Terms deſtroy each other, I conclude that $5\,a + 3\,b$ is the Root required.

## XXIII.

To render the Method of extracting the ſquare Root familiar to Beginners, here follow more Examples. Another example of the extraction of the ſquare root.

Let it be propoſed to extract the ſquare Root of the Quantity $4\,a^2 - 4\,a\,b + 4\,a\,c + b^2 - 2\,c\,b + c^2$, whoſe Terms are ranged according to the Dimenſions of the Letter $a$.

$$\begin{array}{r|l} 4\,a^2 - 4\,a\,b + 4\,a\,c + b^2 - 2\,b\,c + c^2 & 4\,a - 2\,b + c \\ -\,4\,a^2 & 2\,a - b + c \\ & 4\,a - b \\ \hline -\,4\,b\,a + 4\,c\,a + b^2 - 2\,c\,b + c^2 & \\ +\,4\,b\,a - b^2 & \\ \hline 4\,c\,a - 2\,c\,b + c^2 & \\ -\,4\,c\,a + 2\,c\,b - c^2 & \\ \hline 0 & \end{array}$$

*Example* 3.

$$\begin{array}{r|l}
9x^4 - 24a^2x^2 + 16a^4 & 6x^2 - 4a^2 + 2b^2 \\
+ 12b^2x^2 - 16a^2b^2 & \\
- 9x^4 + 4b^4 & 3x^2 - 4a^2 + 2b^2 \\
\hline
- 24a^2x^2 + 16a^4 & \\
+ 12b^2x^2 - 16a^2b^2 & \\
+ 4b^4 & \\
+ 24a^2x^2 - 16a^4 & \\
- 12b^2x^2 + 16a^2b^2 & \\
- 4b^4 & \\
\hline
0 &
\end{array}$$

*Example* 4.

$$\begin{array}{r|l}
y^4 + 4y^3 \bullet - 8y + 4 & 2y^2 + 4y - 2 \\
- y^4 & y^2 + 2y - 2 \\
\hline
+ 4y^3 \bullet - 8y + 4 & 2y^2 + 2y \\
- 4y^3 - 4y^2 & \\
\hline
- 4y^2 - 8y + 4 & \\
+ 4y^2 + 8y - 4 & \\
\hline
0 &
\end{array}$$

XXIV.

Method of extracting the ſquare root of numbers.

The ſquare Root of any Number may be found out after the ſame Manner. If it be a Number under 100, its neareſt ſquare Root is found by the following Table.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Squares | 1, | 4, | 9, | 16, | 25, | 36, | 49, | 64, | 81, | 100. |
| Roots | 1, | 2, | 3, | 4, | 5, | 6, | 7, | 8, | 9, | 10. |

Where it is eaſy to obſerve, that the Square of a ſimple Number cannot conſiſt of more than two Places of Figures, ſince the Square of 10 the leaſt Number conſiſting of two Places is 100, the leaſt Number conſiſting of three Places. That the Square of a Number conſiſting of two Places, cannot conſiſt of more than four Places; for the Square of 100, the leaſt Number conſiſting of three Places, is 10000, the leaſt Number conſiſting of five Places; and in general, that the Square of any Number cannot conſiſt of more than the Double of its Places.

From whence it follows that a Number conſiſting of leſs than three Places, can have but one Figure in its Root; that a Number conſiſting of more than three, but leſs than five Places, can have but two Figures in its Root; that a Number conſiſting of more than five, but leſs than ſeven Places, can have but three Figures in its Root, and ſo on, taking for Limits the odd Numbers 1, 3, 5, 7, 9, 11, &c. whoſe common

Difference is 2. Consequently the Number of Places which the square Root of any Number consists of, is discoverable by Inspection. How the Numbers corresponding to those Places are found, we shall now explain.

XXV.

Let it be proposed to extract the square Root of 99856, as it consists of five Places, I conclude that the Root will consist of three. To determine the Numbers corresponding to those Places, I express them by $x$, $y$ and $z$, consequently $x^2 + 2xy + y^2 + (2x + 2y)z + z^2 = 99856$. Hence the Difficulty is reduced to find the Terms of this Quantity in the given Number 99856. First example.

The first Number $x$ being Hundreds, its Square $x^2$ will be Tens of Thousands, or Tens of Thousands with Hundreds of Thousands; consequently this Term should be contained in 9, the Number corresponding to the fifth Place of the Proposed, and its square Root 3 (Hundreds) is the first Figure of the Root, which I place in the Quotient, and subtract the Square of 3 (Hundreds) from the proposed Number.

The Number $y$ being Tens and multiplied by $2x$, the Double of the first, the Product should be Thousands or Thousands with Tens of Thousands, consequently the Term $2xy$ should be contained in 9, the Number corresponding to the fourth Place of the Proposed. I divide therefore 9 by 6, the Quotient is 1 (Ten) which being the second Part of the Root, I place it after the first Figure 3, and as $y^2$ the Square of the second Part of the Root is Hundreds, I conclude that the Terms $2xy + y^2$ should be contained in 98, the Numbers corresponding to the third and fourth Places of the Proposed. I therefore multiply 6 Hundreds and 1 Ten or 61 by 1, and subtract the Product from 98, and the Remainder is 37 (Hundreds).

It remains to find in the given Number the Terms $2xz + 2yz$, the Product of double the first Part of the Root (Hundreds) into the last (Units) which will produce either Hundreds or Tens of Hundreds, and the Product of the Double of the second Part of the Root (1 Ten) into the last (Units), from whence can arise only Tens or Tens with Hundreds, they will therefore be contained in 375, omitting the 6 Units which does not affect those two Products. I divide therefore 375 by 62, the Double of the first and second Figures of the Root, and the Quotient is 6, which being the third Figure of the Root, I place it after the second (1), and as $z^2$ the Square of this third Figure is Units or Units with Tens, I conclude that 3756 should contain the Terms $(2x + 2y)z + z^2$, the Product of Double of the first and second Figures of the Root into the third, together with the Square of the third; I multiply therefore 626 by 6, and subtract the Product from 3756, and as there nothing remains, I conclude that 316 is the Root required.

```
9·98·56 (316
9
---
61 ) 98
×1 ) 61
-----
626 ) 3756
×6 ) 3756
-----
     0
```

XXVI.

Process of the extraction of the square root of numbers.

It is easy to perceive that the square Root of any Number may be discovered by the same Method of Reasoning, the Process of which may be expressed thus. Put a Point in the Place of the Tens, and omitting one, point every other Figure towards the left Hand, and by these Points the Number will be distinguished into as many Periods as there are Figures in the Root. Then find the square Root of the first Period, and it will give the first Figure of the Root, subtract its Square from that Period, and annex the second Period of the given Number to the Remainder, then divide this new Number (neglecting its last Figure) by the Double of the first Figure of the Root, and quote the Number of Times, then annexing the Quotient to that Double, multiply the Number thence arising by the said Quotient, and subtract the Product from the whole Dividend, as before. Annex the third Part of the given Number to the Remainder, which is to be managed exactly as the last, and proceed thus until all the Periods are brought down. If at last there be no Remainder, then will the Quotient express the true Root.

XXVII.

Second example.

Let the square Root of 27394756 be required. I first point it into Periods of two Figures each, and as there are four Periods, I conclude there will be four Places in the Root, then I find the nearest square Root of 27 to be 5, which therefore is the first Figure of the Root. I subtract 25, the Square of 5 from 27, and to the Remainder 2, I annex the second Period 39, and I divide (neglecting the last Figure 9) by the Double of 5 or by 10, and I place the Quotient after 5, and then multiply 102 by 2, and subtract the Product 204 from 239; then to the Remainder 35, I annex the third Period 47, and dividing 3547 (neglecting the last Figure 7) by the Double of 52, that is, by 104, I place the Quotient after 2, and multiplying 1043 by the Quotient 3, I find the Product to be 3129, which I subtract from the Dividend 3547, and to the Remainder 418 I annex the last Period 56, and dividing 41856 (neglecting the last Figure 6) by the Double of 523, that is, by 1046 I place the Quotient after 3, and multiplying 10464 by the Quotient 4, I find the Product to be 41856, which subtracted from the Dividend and leaving no Remainder, the exact Root must be 5234.

27·39·47·56 (5234
25
102 ) 239
×2 ) 204
1043 ) 3547
×3 ) 3129
10464 ) 41856
×4 ) 41856

XXVIII.

Third example.

Let the ſquare Root of 389489 be required. After having firſt Pointed it, I then find the neareſt ſquare Root of 38 to be 6 which therefore is the firſt Figure of the Root; I ſubtract 36, the Square of 6, from 38, and to the Remainder I annex the ſecond Period 94, and I divide (neglecting the laſt Figure 4) by the double of 6 or by 12 and I place the Quotient after 6, and then multiply 122 by 2, and ſubtract the Product 244 from 294. then to the Remainder (50) I annex the third Period 89, and dividing 5089 (neglecting the laſt Figure 9) by the Double of 62, or by 124, I place the Quotient after 2, and Multiplying 1244 by the Quotient 4, I find the Product to be 5976, which ſubſtracted from the Dividend leaves the Remainder 113, and as there are no more Periods to bring down there are to be no more Figures in the Root; becauſe a Number conſiſting of ſix Places cannot have more than three Figures in its Root; but as 624 Multiplyed by Itſelf gives for Product 389376, which is leſs than the propoſed Number 389489, and that 625 multiplyed by Itſelf gives for Product 390625, which is greater than the propoſed Number, I conclude that 624 is the neareſt Root in Integers.

```
38.94.89.(624.
  36
122 )294
 ×2 )244
1244 )5089
 ×4 )4976
      113
```

XXIX.

The root of a number which is not a perfect power of the ſame degree as the root required is not determinable.

As this Reaſoning may be applied to any other Example whatſoever, it appears in general, that the ſquare Root $a$ of any Number $A$, found by the foregoing Method, when there is a Remainder left, is leſs than $a+1$ but greater than $a$; conſequently if it be determinable it will be $a+\frac{m}{n}$, $\frac{m}{n}$ expreſſing any proper Fraction reduced to its leaſt Terms.

Now ſince $\frac{m}{n}$ is a Fraction reduced to its leaſt Terms, then $\frac{an+m}{n}$ will be alſo a Fraction reduced to its leaſt Terms; for if $n$ and $an+m$ have a common Diviſor, then the Denominator of the Fraction $\frac{an+m}{n}$ when reduced would be leſs than $n$, conſequently the Fraction $\frac{m}{n}$ has not been reduced to its leaſt Terms, which is againſt the Suppoſition. if therefore the ſquare Root of a Number which is not an exact Square is determinable. A Fraction reduced to its leaſt Terms when ſquared muſt give an Integer, or which comes to the

ſame Thing, two Numbers $a$ and $b$ that have no common Diviſor and two Others $c$ and $d$ that have no common Diviſor with one another or with the two Firſt, in their Product $\frac{a\,c}{b\,d}$, $a\,c$ and $b\,d$ would have a common Diviſor.

But it is Eaſy to perceive 1° that $a\,c$ and $b$ can have no common Diviſor or $\frac{a\,c}{b}$ cannot be reduced, for if poſſible let $\frac{a\,c}{b} = \frac{e\,b}{g\,b}$ ſince $b = g\,b$, $b$ is a Diviſor of $b$, but $b$ has no common Diviſor with $a$ or $c$ wherefore neither has $b$ for if $b = r\,s$ and, for Example, $a = t\,s$, then $b = g\,r\,s$, that is $a$, and $b$ would have a common Diviſor which is againſt the Suppoſition, Again ſince $a\,c = e\,b$, $\frac{b}{a} = \frac{c}{e}$ but $\frac{b}{a}$ is a Fraction reduced to its leaſt Terms, wherefore $b$ is a Diviſor of $c$, and conſequently $b$ and $c$ have a common Diviſor which is likewiſe againſt the Suppoſition. In like manner it will appear 2° that $a\,c$ and $d$ can have no common Diviſor, and in general that any two Numbers which have no common Diviſor with a third Number, their Product will have no common Diviſor with this third Number, wherefore ſince $b$ and $d$ have no common Diviſor with $a\,c$ their Product $b\,d$ can have no common Diviſor with $a\,c$. hence we may conclude that if $\frac{a}{b}$ is a Fraction reduced to its leaſt Terms $\frac{a^2}{b^2}$, $\frac{a^3}{b^3}$ and in general $\frac{a^n}{b^n}$ will be a Fraction reduced to its leaſt Terms, wherefore a Fraction either pure or mixed raiſed to any Power will give always a Fraction, wherefore if the ſquare Root, cube Root &c. of an integer Number is not an Integer, it cannot be determined: as alſo the Root of a Fraction or of an Integer joined to a Fraction which when reduced to a Fraction each of its Terms is not a perfect Power of the ſame Degree as the Root required, for each Term of the Fraction may be conſidered as an Integer a part, whoſe Root cannot be determined.

## XXX.

Method of approximating to the ſquare root of numbers.

Though the exact Root of an Integer which is not a perfect Square cannot be found, however we may Approximate to it to any degree of Exactneſs, for Example, to extract the ſquare Root of 389489, inſtead of 389489 we may write $\frac{3894890000}{10000}$ which is equal to it, obſerving that the Denominator 10000 be a ſquare Number, that is, ſhould contain an even number of Cyphers then extracting the ſquare Root of the Numerator which may be found true to an Unit, and dividing this Root by 100 which is the ſquare Root of the Denominator, the ſquare Root of

$\frac{389489000}{10000}$, or of 389489, will be obtained true to $\frac{1}{100}$, if we had wrote $\frac{389489000000}{1000000}$ it is easy to perceive that the Root would be found true to $\frac{1}{1000}$, and so on. Hence in extracting the square Root after the Number proposed is gone through, if there is a Remainder the Operation being continued by adding periods of Cyphers to that Remainder, the true Root will be obtained in Decimals to any degree of Exactness.

XXXI.

To Approximate, for Example, to the Root of 389489, having found its nearest Root in Integers, I annex to the Remainder 113 a period of Cyphers, I divide (neglecting the last Figure) by the double of 624 or by 1248, and place the Quotient (0) after 4 and then multiply 12480 by 0 and Subtract (0) from 11300 then to the Remainder (11300) I annex another Period of Cyphers, and dividing 1130000 (neglecting the last Figure) by the double of 6240 or by 12480 I place the Quotient after 0, and multiplying 124809 by the Quotient 9 I subtract the Product 1123281 from 1130000, and to the Remainder 6719 I annex another Period which I manage as the last, and proceeding thus it is Manifest that the Root may be Approximated to any degree of Exactness. If we neglect the Remainder 4780975, the Root will be $\frac{6240905}{10000}$ which is true to $\frac{1}{10000}$, for if instead of $\frac{6240905}{10000}$ we take $\frac{6240906}{10000}$ the Root would be too great, since this Quantity squared gives $\frac{38948907700836}{100000000}$ or $389489 + \frac{7700836}{100000000}$ which is greater than 389489.

First Example.

```
38.94.89(624,0905
36
122)294
 x2)244
1244)5089
  x4 4976
12480)  11300
  x0)     0
124809)1130000
  x9)1123281
1248180)  671900
  x0)       0
1248180 5)67190000
  x5)  624090 25
                4780975
```

XXXII.

After the same way the square Root of decimal Numbers, may be extracted, but before the Operation the Number of decimal Places are to be made even, and after the Operation as many places are to be pointed off in

Second example.

the Root as there are Periods in the Fractional part of the propoſed Number. Thus if the ſquare Root of 3297,6 be required True to four decimal Places, inſtead of 3297,6 I write $\frac{32976000000}{10000000}$ which is equal to it, whoſe Root is 57,4247 in like manner to extract the ſquare Root of 0,99856 true to ſix decimal places, inſtead of 0,99856 I write $\frac{998560000000}{1000000000000}$ which is equal to it, whoſe Root is 0,999279 as will appear by the following Diagrams.

```
32.97;60(57,4247          0,99.85.60(0,999279
 25                            81
107)797                   189)1885
 ×7 749                    ×9 1701
1144)4860                 1989)18460
 ×4 4576                   ×9 17901
11482)28400               19982)55900
 ×2 22964                  ×2 39964
114844)543600             199847)1593600
 ×4 459376                 ×7 1398929
1148487)8422400           1998549)19467100
 ×7 8039409                ×9 17986941
   382991 &c.                1480159
```

XXXIII.

Anether method of approximating to the ſquare root of numbers.

Beſides the foregoing Method for Approximating to the ſquare Root of Numbers, the Analyſts have imagined others which we ſhall proceed to explain.

Let it be propoſed to Approximate to the ſquare Root of 50, the neareſt leſs Root of 50 in Integers is 7, let $7 + z$ be the true Root, wherefore $49 + 14z + z^2 = 50$, or $-1 + 14z + z^2 = 0$: let this Quantity be multiplied by $1 + Az + Bz^2$, the Product will be $-1 + (14 - A)z + (1 + 14A - B)z^2 + (A + 14B)z^3 + Bz^4 = 0$ in which the Terms $(1 + 14A - B)z^2$ and $(A + 14B)z^3$ may be made to vaniſh by determining $A$ and $B$ by means of the Equations $1 + 14A - B = 0$ and $A + 14B = 0$, from whence reſults $A = -\frac{14}{197}$ and $B = -\frac{A}{14} = \frac{1}{197}$, ſubſtituting thoſe Values in the foregoing Equation, we will have $-1 + (14 + \frac{14}{197})z + \frac{z^4}{197} = 0$ which by rejecting the exceeding ſmall Quantity $\frac{z^4}{197}$ becomes $-1 + (14 + \frac{14}{197})z = 0$; hence $-197 + 2772z = 0$, and

$z = \frac{197}{2772} = ,0710678$, which Value is true to the laſt decimal Place, and if more terms of the Series $1 + Az + Bz^2 + Cz^3$ &c. had been taken, the Root would be Exacter in Proportion.

In general to Approximate to the ſquare Root of any Number $A$ let $r$ be the neareſt leſs Root in Integers, and $r + z$ the true Root, then will $r^2 + 2rz + z^2 = A$, $2rz + z^2 = A - r^2$, or $-\frac{m}{2r} + z + \frac{z^2}{2r} = 0$ (putting $A - r^2 = m$) then multiplying by $1 + Az$ there will reſult

$$-\frac{m}{2r} + \left(1 - \frac{mA}{2r}\right)z + \left(\frac{1}{2r} + A\right)z^2 + \frac{A}{2r}z^3 = 0,$$

here by making $\frac{1}{2r} + A = 0$, $A$ is found $= -\frac{1}{2r}$, and our Equation becomes $-\frac{m}{2r} + \left(1 + \frac{m}{4r^2}\right)z - \frac{1}{4r^2}z^3 = 0$, which by neglecting the laſt Term as being very ſmall, will be reduced to $-\frac{m}{2r} + \left(1 + \frac{m}{4r}\right)z = 0$, whence $z = \frac{rm}{2r^2 + \frac{1}{2}m}$.

To obtain a more approximate Value of $z$, we have only to multiply $-\frac{m}{2r} + z + \frac{z^2}{2r} = 0$ by $1 + Az + Bz^2$ whence there ariſes

$$-\frac{m}{2r} + \left(1 - \frac{mA}{2r}\right)z + \left(\frac{1}{2r} + A - \frac{mB}{2r}\right)z^2 + \left(\frac{A}{2r} + B\right)z^3 + \frac{B}{2r}z^4 = 0$$

in which the third and fourth Terms will vaniſh by determining $A$ and $B$ by means of the Equations, $\frac{1}{2r} + A - \frac{mB}{2r} = 0$ & $\frac{A}{2r} + B = 0$, from whence $A$ is found $= -\frac{2r}{4r^2 + m}$ and $B = -\frac{A}{2r} = \frac{1}{4r^2 + m}$; wherefore $-\frac{m}{2r} + \left(\frac{2r^2 + m}{2r^2 + \frac{1}{2}m}\right)z + \frac{1}{2r(4r^2 + m)}z^4 = 0$; which by rejecting the laſt Term as exceeding ſmall, will become

$$-\frac{m}{2r} + \left(\frac{2r^2 + m}{2r^2 + \frac{1}{2}m}\right)z = 0 \text{ and } z = \frac{m}{2r} \times \frac{2r^2 + \frac{1}{2}m}{2r^2 + m}.$$

To obtain ſtill a more Approximate Value of $z$, it ſuffices to multiply $-\frac{m}{2r} + z + \frac{z^2}{2r}$ by $1 + Az + Bz^2 + Cz^3$, & there reſults

$$-\frac{m}{2r} + \left(1 - \frac{mA}{2r}\right)z + \left(\frac{1}{2r} + A - \frac{mB}{2r}\right)z^2 + \left(\frac{A}{2r} + B - \frac{mC}{2r}\right)z^3$$
$$+ \left(\frac{B}{2r} + C\right)z^4 + \frac{C}{2r}z^5.$$

in which Equation the third, fourth, and fifth Terms may be made to vaniſh by determining $A$, $B$, and $C$ by means of the Equations

$\frac{1}{2r} + A - \frac{mB}{2r} = 0$, $\frac{A}{2r} + B - \frac{mC}{2r} = 0$, $\frac{B}{2r} + C = 0$

whence $B = \frac{2rA+1}{m}$, $C = \frac{2rB+A}{m}$ and $C = -\frac{B}{2r}$,

exterminating $C$, there ariſes $\frac{2rB+A}{m} + \frac{B}{2r} = 0$, which by ſubſtituting the value of $B$, becomes $\frac{4r^2A+2r}{m^2} + \frac{2rA+1}{2rm} + \frac{A}{m} = 0$, wherefore $A = -\frac{4r^2+m}{8r^3+4rm}$, and $C = -\frac{1}{8r^3+4rm}$, and of courſe

$-\frac{m}{2r} + \left(\frac{16r^4+12r^2m+m^2}{16r^4+8r^2m}\right)z - \frac{1}{16r^4+8r^2m}z^3 = 0$,

from which, rejecting the laſt Term as exceeding ſmall, you will find

$z = \frac{8r^3m+4rm^2}{16r^4+12r^2m+m^2} = \frac{rm(2r^2+m)}{r^2(4r^2+3m)+\frac{1}{4}m^2}$.

XXXIV.

To ſhew the Uſe and great Exactneſs of theſe Approximations by an Example, let it be propoſed to extract the ſquare Root of 441.

First example.

Suppoſing $r$ to be aſſumed $= 20$, then $m = 41$, and $z = \frac{1640}{1641}$ & conſequently $r + z = 21 - \frac{1}{1641}$ by the firſt Approximation.

By the Second $r + z = 20 + \frac{67281}{67280} = 21 + \frac{1}{2758481}$ more nearly.

And by the Third $r + z = 20 + \frac{2758480}{2758481} = 21 - \frac{1}{2758481}$, ſtill more nearly.

Second Example.

Let it be propoſed to extract the ſquare Root of 108, here $r = 10$, $m = 8$, and $z = \frac{80}{204} = ,392$. Hence the Root required is 10 ,392 nearly, which value is true to the laſt decimal Place.

By the Second $r + z = 10 + \frac{1632}{4160} = 10,39230$ more nearly.

And by the Third $r + z = 10 + \frac{16640}{42416} = 10,3923048$, true to the laſt decimal Place.

It is to be obſerved that a vulgar Fraction cannot always rigourouſly and exactly be reduced to a Decimal. let $\frac{p}{q}$, for Example, be propoſed to be reduced to a Decimal $\frac{r}{10^n}$, then $r = \frac{p \times 10^n}{q}$ but

$10^n = 2^n \times 5^n$, now $\frac{p \times 2^n \times 5^n}{q}$ cannot give an Integer $r$, unleſs $q$ be ſome Power of 2, or of 5, or of $2 \times 5$, leſs than $n$, for $\frac{p}{q}$ is ſuppoſed to be a Fraction reduced to its leaſt Terms, in every other Caſe $\frac{p \times 10^n}{q}$ can never give an Integer $r$, but it is manifeſt the greater $n$ is, the nearer $\frac{r}{10^n}$ will approach to $\frac{p}{q}$, the Error being always leſs than $\frac{1}{10^n}$, ſince dividing $p \times 10^n$ by $q$ the Quotient which reſults, and which is too little will be too great if increaſed by an Unit, wherefore $\frac{r}{10^n} < \frac{p}{q}$ and $\frac{r+1}{10^n} > \frac{p}{q}$.

A vulgar fraction cannot always be reduced to a decimal.

XXXV.

To ſhew the uſe of the foregoing Rules, in the Solution of Problems, here follow ſome Examples.

Problem.

*A Gentleman left an Eſtate of* r *Pounds per Annum to his Son, who being a Minor, his Guardian allowed him a Sum* a *the firſt Year, and puts out at Intereſt the Remainder* r — a. *He allowed him the ſecond Year a Sum* b, *and the over-plus* r — b *was alſo put out at Intereſt, and the Guardianſhip being ended a few Days after, the Revenue of the young Gentleman was found to be Increaſed during the two Year's Guardianſhip by a Sum* d. *What rate of Intereſt was allow'd.*

Let $\frac{100}{x}$ expreſs the yearly Intereſt of 100*l.* I ſay 100 : $\frac{100}{x} = r - a$ : $\frac{r-a}{x}$ Sum added to the Revenue of the young Gentleman at the End of the firſt Year, wherefore the Sum ſaved at the End of the firſt Year will be $r - a + \frac{r-a}{x}$, the Intereſt of which Sum at the End of the Second Year will be $\frac{r-a}{x} + \frac{r-a}{x^2}$, which together with $\frac{r-b}{x}$ the Intereſt of $r - b$ put out at Intereſt at the beginning of the ſecond Year will expreſs the Increaſe, which the Revenue of the young Gentleman has received during the two year's Guardianſhip. Conſequently $\frac{2r-a-b}{x} + \frac{r-a}{x^2} = d$, putting $2r - a - b = m$, $r - a = c$ (to abridge the Computation) there reſults the Equation $\frac{m}{x} + \frac{c}{x^2} = d$, or $x^2 + \frac{mx}{d} = \frac{c}{d}$, and compleating the Square and extracting the

ſquare Root $x = \frac{m}{2d} \pm \sqrt{\left(\frac{c}{d} + \frac{m^2}{4d^2}\right)} = \frac{m \pm \sqrt{4cd + m^2}}{2d}$

XXXVI.

Application of the foregoing Problem to an example.

To apply this general Solution to an Example, let the Eſtate be 4800 *l.* per Annum, and ſuppoſe the young Gentleman to have expended the firſt Year 2400*l.* and the ſecond Year 3120*l.* and let the increaſe of his Revenue at the End of the ſecond Year be found to be 356*l.* 13*s.* 4*d.*

We will have $r = 4800$, $a = 2400$, $b = 3120$, $d = \frac{1070}{3}$ $2r - a - b = m = 4080$, Subſtituting thoſe Values in the Expreſſion $x = \frac{m \pm \sqrt{(4cd + m^2}}{2d}$ we will have $x = \frac{3[4080 \pm \sqrt{(20070400)}]}{2140}$ $= \frac{3(4080 \pm 4480)}{2140}$, wherefore $x = 12$ and Conſequently the rate of Intereſt allowed was 8*l.* 6*s.* 8*d.* per Cent. in Effect the Intereſt of 2400*l.* for one Year at 8*l.* 6*s.* 8*d.* per Cent. amounts to 200*l.* conſequently the Sum ſaved at the End of the firſt Year will be 2600*l.* the intereſt of which for one Year at 8*l.* 6*s.* 8*d.* per Cent. will be 216*l.* 13*s.* 4*d.* and the Intereſt of 1680*l.* at 8*l.* 6*s.* 8*d.* for one Year is 140*l.* and conſequently the increaſe of the Revenue of the young Gentleman during the two Years Guardianſhip will be 356*l.* 13*s.* 4*d.* agreeable to the Conditions of the Problem.

If the Revenue of the young Gentleman was ſuppoſed to be increaſed 432*l.* during the two Years Guardianſhip, then $x$ will be found $= 10$ and conſequently the rate of Intereſt allowed will be 10 per Cent. in effect the Intereſt of 2400*l.* for one Year at 10 per Cent. will be 240*l.* the Sum ſaved therefore, at the End of the firſt Year will be 2640*l.* the Intereſt of which for one Year at 10 per Cent. will be 264*l.* and the Intereſt of 1680*l.* for one Year at 10 per Cent. will be 168*l.* wherefore the increaſe of the Revenue of the young Gentleman during the two Years Guardianſhip will be 432*l.* as the Conditions of the Problem require.

XXXVII.

Another Problem.

*Two Notes one of* 120*l. payable in* 6 *Months and the other of* 150*l. payable in* 9 *Months, were diſcounted for* 81*l.* 10*s. what rate of Intereſt were they diſcounted at.*

Let $x$ denote the Intereſt of one Pound for 12 Months, then the amount of 1*l.* in 6 Months being $1 + \frac{x}{2}$ and in 9 Months $1 + \frac{3x}{4}$ the preſent Value of the Bill due at the end of the 6 Months will therefore be $\frac{120}{1 + \frac{1}{2}x}$, and that of the Bill due at the end of 9 Months $\frac{150}{1 + \frac{3}{4}x}$.

whence we have $\frac{120}{1+\frac{1}{2}x} + \frac{150}{1+\frac{1}{4}x} = 120 + 150 - 8,5 = 261,5$ by the Queſtion, which, by Reduction, becomes $120 + 90x + 150 + 75x = 261,5 \times (1 + \frac{3}{8}x + \frac{3}{8}x^2)$ or, $270 + 165x = \frac{261,5 \times (8 + 10x + 3x^2)}{8}$, therefore $\frac{(270 + 165x) \times 8}{261,5} = 3x^2 + 10x + 8$, that is, $\frac{2160}{261,5} + \frac{1320}{261,5}x = 3x^2 + 10x + 8$, from which we have $x^2 + \frac{2590}{3 \times 523}x = \frac{136}{3 \times 523}$, whence by compleating the Square, &c. $x$ will be found $= \sqrt{\left[\frac{136}{1569} + \left(\frac{1295}{1569}\right)^2\right]} - \frac{1295}{1569} = \frac{\sqrt{(136 \times 1569 + 1295 \times 1295)} - 1295}{1569} = 0,05093$, which multiplied by 100 gives £5, 093, or £5. 1*s.* 10*d.* $\frac{1}{2}$. nearly, for the Rate per Cent. at which the Notes were diſcounted.

XXXVIII.

In the different Examples which we have given of the Reſolution of Equations of the ſecond Degree, the young Analyſt could ſcarce meet with any Difficulty except when it was propoſed to reduce radical Quantities by taking from under the Sign, the ſquare Quantities which were Factors of the radical Quantity; In Effect this Operation is the niceſt that occurs in the Reſolution of Equations of the ſecond Degree; to aſſiſt therefore the young Analyſt to perform it with Facility, here follow ſome Examples.

Examples of the reduction of radical quantities.

$$\sqrt{48aabc} = 4a\sqrt{3bc}, \sqrt{\frac{(a^3b + 4aabb + 4ab^3)}{cc}} = \frac{(a + 2b\sqrt{ab})}{c},$$

$$\sqrt{\left(\frac{aabbmm}{ppxx} + \frac{4aam^3}{pxx}\right)} = \frac{am}{px}\sqrt{(bb + 4mp)},$$

$$6\sqrt{\frac{75}{98}}abb = \frac{30b}{7}\sqrt{\frac{3a}{2}}.$$

XXXIX.

No ſooner had Equations of the ſecond Degree made known the irrational or incommenſurable Quantities (it is thus Quantities which have no exact Root are called) but the Analyſts found it neceſſary to perform on thoſe Quantities the ſame Operations as on rational or commenſurable Quantities, that is, they had to add, ſubtract, multiply, and divide Quantities either entirely incommenſurable, or partly incommenſurable and partly commenſurable.

Quantities which have not an exact root are called incommenſurable or irrational quantities.

As to the Addition and Subtraction of radical Quantities, the only Difficulty conſiſts in reducing them to their leaſt Expreſſions.

The addition and ſubtraction of

irrational quantities presupposes only their reduction to their least terms.

For Example, Let it be proposed to add $\sqrt{48\,a\,b\,b}$ and $b\sqrt{75\,a}$ together, I reduce the first of those Quantities to $4\,b\sqrt{3\,a}$, and the second to $5\,b\sqrt{3\,a}$, whose Sum is $9\,b\sqrt{3\,a}$.

In like Manner $\sqrt{48\,c^3} - \sqrt{\frac{16}{27}\,c^3} = \frac{32}{9}\,c\sqrt{3\,c}$,

$$\sqrt{\frac{a\,b^3}{c\,c}} + \frac{1}{2\,c}\sqrt{(a^3\,b - 4\,a\,a\,b\,b + 4\,a\,b^3)} = \frac{a}{2\,c}\sqrt{a\,b}.$$

$$a\sqrt{\left(\frac{a^3\,b}{3\,a\,a + 6\,a\,c + 3\,c\,c}\right)} - \frac{b\,c\sqrt{a\,b}}{a+c} = \left(\frac{a\,a}{\sqrt{3}} - b\,c\right) \times \frac{\sqrt{a\,b}}{a+c}.$$

XL.

Multiplication of Surds.

With Respect to Multiplication, if the Quantities to be multiplied are both incommensurable, there is no more to be done than to multiply the Quantities which are under the radical Sign, and set the common radical Sign over the Products; which afterwards is to be reduced to its least Expression.

Let it be proposed for Example, to multiply $\sqrt{a\,b}$ by $\sqrt{a\,c}$, I write down $\sqrt{a\,a\,b\,c}$ or $a\sqrt{b\,c}$. In like Manner

$\sqrt{3\,c\,d} \times \sqrt{4\,f\,g\,c} = \sqrt{12\,f\,g\,c\,c\,d} = 2\,c\sqrt{3\,f\,g\,d}$.

When the radical Quantities to be multiplied are equal, the Multiplication is performed by taking away the radical Sign. The Product, for Example, of $\sqrt{a^3\,c\,d}$ into $\sqrt{a^3\,c\,d}$ is $a^3\,c\,d$.

If the Quantity by which the Surd is to be multiplied be rational, it suffices to write it before the Sign between Parentheses when it consists of several Terms; and before it can be brought under the Sign it must be first squared.

For Example, the Product of $a + b$ into $\sqrt{\frac{f\,f\,g}{a\,a - b\,b}}$ is

$$(a+b)\sqrt{\frac{f\,f\,g}{a\,a-b\,b}}, \text{ or } \sqrt{\left[\frac{f\,f\,g \times (a\,a + 2\,a\,b + b\,b)}{a\,a - b\,b}\right]}$$

$$\text{or } \sqrt{\left[\frac{f\,f\,g \times (a+b)}{a-b}\right]}, \text{ or } \frac{f\sqrt{(a\,g + b\,g)}}{\sqrt{(a-b)}},$$

$$2\,a\sqrt{\frac{a\,a + b\,b}{c}} \times -3\,a\sqrt{\frac{(a\,a + b\,b)}{d}} =$$

$$-\frac{6\,a\,a \times (a\,a + b\,b)}{\sqrt{c\,d}} = -\frac{6\,a^4 + 6\,a\,a\,b\,b}{\sqrt{c\,d}}.$$

$\sqrt{(a+b)} \times \sqrt{(a-b)} = \sqrt{(a\,a - b\,b)}$.

If the Quantities to be multiplied consist of several Parts, either all irrational or partly irrational and partly rational, the Operation is performed by observing the same Rules as in the Multiplication of compound Quantities, for Example, $(3\,a\sqrt{b\,c} - 2\,b\sqrt{a\,c}) \times 2\,c\sqrt{a\,b} = 6\,a\,b\,c\sqrt{a\,c} - 4\,a\,b\,c\sqrt{b\,c}$. $[\sqrt{a\,b} + \sqrt{(a\,a - x\,x)}] \times [\sqrt{a\,b} + \sqrt{(a\,a - x\,x)}] = a\,a + a\,b - x\,x + 2\sqrt{(a^3\,b - a\,b\,x\,x)}$. $[a + \sqrt{(a\,a - b\,b)}] \times [a + \sqrt{(a\,a - b\,b)}] = 2\,a\,a - b\,b + 2\,a\sqrt{(a\,a - b\,b)}$, $[a + \sqrt{(a\,a - x\,x)}] \times [a - \sqrt{(a\,a - x\,x)}]$

$$= xx, \frac{2ax}{c} - \sqrt{\frac{a^3}{c}} \times 3a + \sqrt{\frac{abb}{c}} = \frac{6aax}{c}$$
$$- 3a\sqrt{\frac{a^3}{c}} + \frac{2ax}{c}\sqrt{\frac{aab}{c}} - \frac{aab}{c}.$$

XLI.

Division of incommensurable quantities.

Irrational Quantities are divided one by the other, by dividing the Quantities under the Sign and ſetting the Sign over the Quotient.

If an irrational Quantity is to be divided by a rational one, the irrational Quantity is to be placed above a ſmall Line and the rational Quantity under it, and before it can be brought under the Sign it muſt be firſt ſquared.

If the Surds have any rational Coeficients, their Quotient is to be prefixed to the radical Quotient. All which will appear plain by the following Examples.

$$\frac{\sqrt{ab}}{\sqrt{a}} = \sqrt{b}, \quad \frac{ac\sqrt{bc}}{a\sqrt{b}} = c\sqrt{c}, \quad \frac{12ac\sqrt{6bc}}{4c\sqrt{2b}}$$
$$= 3a\sqrt{3c}, \quad \frac{\sqrt{(aa - xx)}}{\sqrt{(a + x)}} = \sqrt{(a - x)}, \quad \frac{\sqrt{aabb - bbxx}}{a - x}$$
$$= b\sqrt{\frac{a + x}{a - x}}.$$

If the Quantities to be divided conſiſt of ſeveral Parts rational and irrational, the Operation will be performed as the Diviſion of compound Quantities.

For Example, $a^4 + b^4$ being divided by $a^2 + ab\sqrt{2} + bb$ the Quotient will be found to be $a^2 - ab\sqrt{2} + b^2$, $a^3b - abbc - aab\sqrt{bc} + bbc\sqrt{bc}$ divided by $a - \sqrt{bc}$ the Quotient is $aab - bbc$. In like Manner, $a^3 - abc + aa\sqrt{bc} - bc\sqrt{bc}$ divided by $a - \sqrt{bc}$ the Quotient will be $aa + bc + 2a\sqrt{bc}$.

XLII.

The foregoing Principles are ſufficient for reſolving Equations of the ſecond Degree when they include only one unknown Quantity; but as there frequently occur Problems where ſeveral unknown Quantities are concerned, we ſhall proceed to explain how they are to be managed; with this View we ſhall propoſe the following Problems.

First Problem of the ſecond degree where two unknown quantities are concerned.

*The Joint-Stock of two Partners* A *and* B *was* £165. (a), *A's Money was in Trade* 12 *Months, and* B's 8 *Months, when they ſhared Stock and Gain,* A *received* £67. (b) *and* B £126. (c) *what was each Man's Stock?*

Suppoſe *A* advanced $x$ Pounds, then *B* advanced $a - x$ Pounds, and their reſpective Shares of the Gain may be conſidered as the Intereſt of their reſpective Capitals during the Time they were in Trade, the Rate of Intereſt being the ſame for both: Let $z$ therefore denote the Intereſt of £1. for 12 Months, then the Amount of £1. for 12 Months being $1 + z$, and in 8 Months $1 + \frac{2}{3}z$, the Dividend of *A* will be $x(1 + z)$,

the Dividend of $B$ $(a - x) \times (1 + \frac{2}{3} z)$, hence $x (1 + z) + (a - x) \times (1 + \frac{2}{3} z) = b + c$, from whence is deduced $z = \frac{3m - 3a}{x + 2a}$ (putting $b + c = m$ to abridge the Computation) substituting this Value of $z$ in $x + z x$, the Dividend of $A$, there will result $x + \frac{3mx - 3ax}{x + 2a} = d$, or $x^2 + 3mx - ax - dx = 2ad$, now putting $3m - a - d = n$ compleating the Square and extracting the Root there will result $x = -\frac{1}{2}n \pm \sqrt{(2ad + \frac{1}{4}n^2)} = 55$.

XLIII.

Second Problem.

*A Grocer sold 80lb. of Mace, and 100lb. of Cloves for £65. he sold 60lb. more of Cloves for £20. than he did of Mace for £10. what Price did he sell each at?*

Suppose he sold the Mace at $x$ and the Cloves at $y$ Shill. a *lb.* then $x$ : 1 *lb.* = £10. or 200*s.* : $\frac{200}{x}$ the Number of Pounds of Mace sold for £10. and $y$ : 1 *lb.* = £20. or 400*s.* : $\frac{400}{y}$ Number of Pounds of Cloves sold for £20. now £65. or $1300 = 80x + 100y$ and $\frac{200}{x} + 60 = \frac{400}{y}$ by the Question. From the first Equation I deduce $\frac{130 - 8x}{10} = y$, and from the second Equation $\frac{20x}{10 + 3x} = y$, wherefore $\frac{130 - 8x}{10} = \frac{20x}{10 + 3x}$, or $1300 - 80x + 390x - 24x^2 = 200x$, that is, $24x^2 - 110x = 1300$, or $x^2 - \frac{110}{24}x = \frac{1300}{24}$, and compleating the Square and extracting the Root, there results $x - \frac{55}{24} = \frac{185}{24}$ or $x = 10$, consequently $y = 5$.

XLIV.

Third Problem.

*The Sum and the Sum of the Squares of three Numbers in geometrical Proportion being given to find the Numbers.*

Let the Sum of the three Numbers sought be denoted by $x$, $y$, $z$, from the Nature of Proportion $x : y = y : z$, that is, $yy = xz$; moreover, since their Sum is given, denoting this Sum by $a$ we will have $x + y + z = a$, and expressing the Sum of their Squares by $bb$ we will have by the last Condition of the Problem $xx + yy + zz = bb$.

To make use of those three Equations, I first exterminate $z$ by Means of its Value $a - x - y$ deduced from the Equation $x + y + z = a$; substituting therefore this Value of $z$ in the two other Equations they will be transformed into $2yy + 2xx + 2xy + aa - 2ax - 2ay$

$= bb$, and $yy = ax - xx - xy$. To exterminate out of those two Equations one of the two unknown Quantities they include, it suffices to find the Value of this unknown Quantity in each of those Equations, and to equate them; let $x$ be the unknown Quantity to be exterminated, the first Equation will give

$$x = -\tfrac{1}{2}y + \tfrac{1}{2}a \pm \sqrt{\left(\frac{bb}{2} - \frac{aa}{4} - \tfrac{3}{4}yy + \tfrac{1}{2}ay\right)},$$

and the second $x = -\tfrac{1}{2}y + \tfrac{1}{2}a \pm \sqrt{(\tfrac{1}{4}a^2 - \tfrac{1}{2}ay - \tfrac{3}{4}y^2)}$. Equating those two Values of $x$, there will result the Equation

$$-\tfrac{1}{2}y + \tfrac{1}{2}a \pm \sqrt{\left(\frac{bb}{2} - \frac{aa}{4} - \tfrac{3}{4}yy + \tfrac{1}{2}ay\right)} = -\tfrac{1}{2}y + \tfrac{1}{2}a \pm \sqrt{(\tfrac{1}{4}aa - \tfrac{1}{2}ay - \tfrac{3}{4}yy)}$$

to solve which I observe first, that the Terms $-\tfrac{1}{2}y + \tfrac{1}{2}a$ being common to both Members of the Equation, the Equation may be reduced to

$$\pm\sqrt{\left(\frac{bb}{2} - \frac{aa}{4} - \tfrac{3}{4}yy + \tfrac{1}{2}ay\right)} = \pm\sqrt{(\tfrac{1}{4}aa - \tfrac{1}{2}ay - \tfrac{3}{4}yy)}$$

the two Members of which being squared, and reduced, there results $ay = \tfrac{1}{2}aa - \tfrac{1}{2}bb$, from whence is deduced $y = \dfrac{aa - bb}{2a}$, this Value of $y$ substituted in one of the foregoing Values of $x$, will give

$$x = \tfrac{1}{4}a + \frac{bb}{4a} \pm \sqrt{\left(\tfrac{1}{4}aa - \tfrac{1}{4}aa + \tfrac{2}{4}bb - \tfrac{3}{16}aa + \tfrac{3}{8}bb - \frac{3b^4}{16aa}\right)},$$

or $x = \dfrac{bb + aa \pm \sqrt{(10bbaa - 3a^4 - 3b^4)}}{4a}$ and substituting the Values of $x$ and $y$ in the Equation $z = a - x - y$ we will have $z = \dfrac{aa + bb \mp \sqrt{(10bbaa - 3a^4 - 3b^4)}}{4a}$.

XLV.

The foregoing equations solved by another Method.

As in this Solution we arrived at a very simple Value of $y$, after having had very complex Radicals, it should incline us to think that this Value might be obtained after a more concise Manner: And it was not difficult to imagine the following Method.

Let the two Equations $x^2 - ax + yx = \dfrac{bb}{2} - \dfrac{aa}{2} - y^2 + ay$ and $x^2 + xy - ax = -yy$ be resumed; subtracting one from the other there results $o = \dfrac{bb}{2} - \dfrac{aa}{2} + ay$, from whence is deduced $y = \dfrac{aa - bb}{2a}$, which being substituted in either one or the other of those two Equations, gives

$$x^2 + \frac{aax - bbx}{2a} - ax = \frac{-a^4 + 2aabb - b^4}{4aa}, \text{ or}$$

the Dividend of $B$ $(a - x) \times (1 + \frac{2}{3} z)$; hence $x (1 + z) + (a - x) \times (1 + \frac{2}{3} z) = b + c$, from whence is deduced $z = \frac{3m - 3a}{x + 2a}$ (putting $b + c = m$ to abridge the Computation) substituting this Value of $z$ in $x + z x$, the Dividend of $A$, there will result $x + \frac{3mx - 3ax}{x + 2a} = d$, or $x^2 + 3mx - ax - dx = 2ad$, now putting $3m - a - d = n$ compleating the Square and extracting the Root there will result $x = -\frac{1}{2}n \pm \sqrt{(2ad + \frac{1}{4}n^2)} = 55$.

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*The Sum and the Sum of the Squares of three Numbers in geometrical Proportion being given to find the Numbers.*

Let the Sum of the three Numbers sought be denoted by $x$, $y$, $z$, from the Nature of Proportion $x : y = y : z$, that is, $yy = xz$; moreover, since their Sum is given, denoting this Sum by $a$ we will have $x + y + z = a$, and expressing the Sum of their Squares by $bb$ we will have by the last Condition of the Problem $xx + yy + zz = bb$.

To make use of those three Equations, I first exterminate $z$ by Means of its Value $a - x - y$ deduced from the Equation $x + y + z = a$; substituting therefore this Value of $z$ in the two other Equations they will be transformed into $2yy + 2xx + 2xy + aa - 2ax - 2ay$

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$$x = -\tfrac{1}{2}y + \tfrac{1}{2}a \pm \sqrt{\left(\frac{bb}{2} - \frac{aa}{4} - \tfrac{3}{4}yy + \tfrac{1}{2}ay\right)},$$

and the second $x = -\tfrac{1}{2}y + \tfrac{1}{2}a \pm \sqrt{(\tfrac{1}{4}a^2 - \tfrac{1}{2}ay - \tfrac{3}{4}y^2)}$. Equating those two Values of $x$, there will result the Equation

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to solve which I observe first, that the Terms $-\tfrac{1}{2}y + \tfrac{1}{2}a$ being common to both Members of the Equation, the Equation may be reduced to

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the two Members of which being squared, and reduced, there results $ay = \tfrac{1}{2}aa - \tfrac{1}{2}bb$, from whence is deduced $y = \dfrac{aa - bb}{2a}$, this Value of $y$ substituted in one of the foregoing Values of $x$, will give

$$x = \tfrac{1}{4}a + \frac{bb}{4a} \pm \sqrt{\left(\tfrac{1}{4}aa - \tfrac{1}{4}aa + \tfrac{3}{4}bb - \tfrac{3}{16}aa + \tfrac{3}{8}bb - \frac{3b^4}{16aa}\right)},$$

or $x = \dfrac{bb + aa \pm \sqrt{(10bbaa - 3a^4 - 3b^4)}}{4a}$ and substituting the Values of $x$ and $y$ in the Equation $z = a - x - y$ we will have $z = \dfrac{aa + bb \mp \sqrt{(10bbaa - 3a^4 - 3b^4)}}{4a}$.

XLV.

The foregoing equations solved by another Method.

As in this Solution we arrived at a very simple Value of $y$, after having had very complex Radicals, it should incline us to think that this Value might be obtained after a more concise Manner: And it was not difficult to imagine the following Method.

Let the two Equations $x^2 - ax + yx = \dfrac{bb}{2} - \dfrac{aa}{2} - y^2 + ay$ and $x^2 + xy - ax = -yy$ be resumed; subtracting one from the other there results $o = \dfrac{bb}{2} - \dfrac{aa}{2} + ay$, from whence is deduced $y = \dfrac{aa - bb}{2a}$, which being substituted in either one or the other of those two Equations, gives

$$x^2 + \frac{aax - bbx}{2a} - ax = \frac{-a^4 + 2aabb - b^4}{4aa}, \text{ or}$$

$x^2 - \frac{(bb + aa)x}{2a} = \frac{-a^4 + 2aabb - b^4}{4aa}$ from whence is deduced the same Value of $x$ as above.

XLVI.

To apply this general Solution to an Example, let the Sum of the three Numbers in continual Proportion be 20, and the Sum of their Squares 140; in this Case $a = 20$, and $bb = 140$, consequently $y = 6\frac{2}{3}$, and $x = 6\frac{2}{3} \pm \sqrt{3\frac{5}{16}}$; Viz. $6\frac{2}{3} + \sqrt{3\frac{5}{16}}$ is the greatest of the three Numbers sought, and $6\frac{2}{3} - \sqrt{3\frac{5}{16}}$ the least. For as the Values of $x$ and $z$ in the foregoing Solution are exactly alike, $x$ denotes ambiguously either of the extreme Numbers, and thence there will come out two Values either of which may be $x$, the other being $z$.

XLVII.

When the values of two unknown quantities in a problem are brought out perfectly alike, how the ambiguity thence arising may be avoided.

When there happens to be such an Affinity or Similitude of the Relation of the two Terms to the other Terms of the Question, that in making use of either of them, their Values would be brought out exactly alike; (which will be easily seen) the Analysts to avoid Ambiguity make use of neither of them, but in their Room chuse some third which shall bear a like Relation to both, as suppose the half Sum or half Difference, or any other Quantity related to both indifferently and without alike.

Thus in the precedent Problem, having found $y = \frac{aa - bb}{2a}$, to avoid the Ambiguity arising from the Substitution of this Value in one of the foregoing Values of $x$, I put $\frac{x - z}{2} = u$, and as $\frac{x + z}{2} = \frac{a - y}{2} = \frac{aa + bb}{4a}$, consequently $x = \frac{aa + bb}{4a} + u$ and $z = \frac{aa + bb}{4a} - u$; but $xz = yy = \frac{a^4 + 2aabb + b^4}{16aa} - uu = \frac{a^4 - 2aabb + b^4}{4aa}$ wherefore $uu = \frac{10bbaa - 3a^4 - 3b^4}{16aa}$, and $u = \frac{\sqrt{(10bbaa - 3a^4 - 3b^4)}}{4a}$. consequently $x = \frac{bb + aa + \sqrt{(10bbaa - 3a^4 - 3b^4)}}{4a}$ and $z = \frac{bb + aa - \sqrt{(10bbaa - 3a^4 - 3b^4)}}{4a}$.

XLVIII.

In this Problem, there occurred Quantities which deſtroyed each other, whereby the Computation was rendered extremely ſimple, but as this ſeldom happens in Equations of the ſecond Degree including ſeveral unknown Quantities, we ſhall proceed to explain the Methods employed by the Analyſts in the more complicated Caſes. Let, for Example, the two Equations $x^2 + ax - 2xy = aa + 2yy$ and $xx - 2ax + xy = 2aa - yy$ be propoſed, the firſt of thoſe Equations gives $x = -\frac{1}{2}a + y \pm \sqrt{(\frac{3}{4}aa - ay + 3yy)}$ and the other gives $x = a - \frac{1}{2}y \pm \sqrt{(3aa - ay - \frac{3}{4}yy)}$, equating thoſe two Values, and reducing, there reſults

$$-\frac{3a}{2} + \frac{3}{2}y \pm \sqrt{(\tfrac{3}{4}aa - ay + 3yy)} = \pm\sqrt{(3aa - ay - \tfrac{3}{4}yy)}.$$

Example of equations of the ſecond degree including two unknown quantities more complicated than the foregoing.

To make the Radicals diſappear in this Equation, I firſt ſquare the two Members and there reſults $\frac{9}{4}yy - \frac{9}{2}ay + \frac{9}{4}aa \pm (3y - 3a)\sqrt{(\frac{3}{4}aa - ay + 3yy)} + \frac{3}{4}aa - ay + 3yy = 3aa - ay - \frac{3}{4}yy$, which ſtill contains a radical Quantity. To make it diſappear I write down this Equation thus,

$-\frac{3}{2}aa + \frac{9}{2}ay - 6yy = \pm(3y - 3a)\sqrt{(\frac{3}{4}aa - ay + 3yy)}$ reducing it and leaving the radical Quantity $\pm(3y + 3a)\sqrt{(\frac{3}{4}aa - ay + 3yy)}$ alone on one Side of the Equation; ſquaring afterwards the two Members $\frac{1}{4}a^4 - \frac{9}{2}a^3y + \frac{105}{4}aayy - 54ay^3 + 36y^4 = (9yy - 18ay + 9aa) \times (\frac{3}{4}aa - ay + 3yy)$ which when reduced becomes $9y^4 + 9ay^3 - 30aayy + 27a^3y = 11a^4$ Equation reſulting from the two foregoing ones, which is to be reſolved to obtain the Solution of the Problem which furniſhed thoſe Equations: From whence it appears that in Equations of the ſecond Degree involving ſeveral unknown Quantities it is not as in Equations of the firſt, in which the reſulting Equation when one of the unknown Quantities is exterminated never riſes to a Degree higher than that of the Equations themſelves.

The reſulting equation when one of the unknown quantities is exterminated.

XLIX.

The Equation which reſults when an unknown Quantity is exterminated out of Equations of the ſecond Degree, may be obtained without being at the Trouble of firſt ſolving them, and afterwards making their Radicals diſappear.

Another method by which the unknown quantity is exterminated in the foregoing example.

That this may appear let the two foregoing Equations be reſumed, and from each be deduced the Value of $xx$, the firſt will give $x^2 = aa + 2yy - ax + 2xy$, the ſecond $x^2 = 2aa - yy + 2ax - xy$. Equating thoſe thoſe two Values, we will have $aa + 2yy + 2xy - ax = 2aa - yy - xy + 2ax$, from whence is deduced $x = \frac{3yy - aa}{3a - 3y}$, which being ſubſtituted in one or other of the

two given Equations, in $xx - 2ax + xy = 2aa - yy$, for Example, gives $\frac{9y^4 - 6aayy + a^4}{(3a - 3y)^2} + \frac{(y - 2a) \times (3yy - aa)}{3a - 3y}$ $= 2aa - yy$ which when reduced, becomes $9y^4 + 9ay^3 - 30aayy + 27a^3y = 11a^4$ the same above.

L.

The foregoing method applied to another example.

Let the two Equations $x^2 + axy + bx = cy^2 + dy + e$ and $x^2 + fxy + gx = by^2 + iy + k$ be given, which are the most complicated of the second Degree involving two unknown Quantities.

Deducing a Value of $xx$ from each of these Equations and equating them, we will have $(a - f) \times xy + (b - g) \times x = (c - b) \times y^2 + (d - i) \times y + e - k$, which, by putting, to abridge the Computation, $a - f = l$; $b - g = m$; $c - b = n$; $d - i = p$; $e - k = q$; will be transformed into $lxy + mx = ny^2 + py + q$. From whence is deduced $x = \frac{ny^2 + py + q}{ly + m}$. Which substituted in one or other of the two given Equations, in the first, for Example, there will result $\frac{(ny^2 + py + q)^2}{(ly + m)^2} + \frac{(ay + b) \times (ny^2 + py + q)}{ly + m}$ $= cy^2 + dy + e$, or $(ny^2 + py + q)^2 + (ay + b) \times (ny^2 + py + q) \times (ly + m) = (cy^2 + dy + e) \times (ly + m)^2$.

If we now perform the Operations which are indicated, reduce and order the Equation, there will result

$$y^4 + \frac{(bln + amn + alp + 2np - 2mlc - l^2d)}{aln + nn - l^2c} y^3$$
$$+ \frac{bmn + qal + pbl + pam + p^2 + 2uq - m^2c - 2mld - el^2}{aln + nn - l^2c} y^2$$
$$+ \frac{blq + amq + bmp + 2pq - m^2d - 2mel}{aln + nn - l^2c} y.$$
$$= \frac{m^2e - q^2 - bmq}{aln + n^2 - l^2c},$$ Equation of the fourth Degree which results from two of the most general Equations of the second.

LI.

Whatever be the dimensions of $y$ in two equations if those of $x$ do not rise higher than two it may be exterminated by the foregoing method.

If we had two Equations such as $xxy + axy = abb$ and $xxyy + ccyx = a^4$, they cannot be considered as Equations of the second Degree, because the unknown Product of $x^2$ into $y$ is of three Dimensions, and that of $x^2$ into $y^2$ is of four Dimensions; however $x$ may be exterminated out of those two Equations by the foregoing Method. That this may appear let $A$ express all the Quantities composed of $y$ and known Quantities which affect $x^2$ to whatever Degree they rise, (or in the Language of the Analysts any Function of $y$) in one of the given Equations; let $B$ express the Function of $y$ which affects $x$ in the same Equation, and $C$ the Quantity which is not affected by any Power of

$x$, that is, let the first Equation be $Ax^2 + Bx = C$, and in like Manner let the second be $A'x^2 + B'x = C'$.

From the first I deduce $x^2 = \frac{C - Bx}{A}$, and from the second $x^2 = \frac{C' - B'x}{A'}$, equating those two Values there results $CA' - BA'x = AC' - AB'x$, from whence is deduced $x = \frac{AC' - CA'}{AB' - BA'}$, which being substituted in the Equation $Ax^2 + Bx = C$, gives $A \times (AC' - CA')^2 + B \times (AB' - BA') \times (AC' - CA') = C \times (AB' - BA')^2$, in which substituting for $A, B, C, A', B', C'$, the Functions of $y$ their Values, the Equation sought will be obtained.

LII.

How the foregoing method is to be applied when the quantity to be exterminated rises to higher dimensions than two.

If the unknown Quantity which is to be exterminated out of two given Equations, rises to higher Dimensions than two, it is easy to perceive that by an Operation similar to the foregoing they may be transformed into two others of lower Dimensions, and by repeating this Operation the unknown Quantity will at length be taken away.

If we had, for Example, $y^3 = xyy + 3x$ and $yy = xx - xy - 3$; to take away $y$ I multiply the latter Equation by $y$, and you have $y^3 = xxy - xyy - 3y$, of as many Dimensions as the former, now by making the Values of $y^3$ equal to one another, I have $xyy + 3x = xxy - xyy - 3y$, where $y$ is depressed to two Dimensions, from whence I deduce $yy = \frac{xxy - 3y - 3x}{2x}$ by this therefore, and the most simple one of the Equations first proposed $yy = xx - xy - 3$, I find $y = \frac{2x^3 - 3x}{3xx - 3}$ which Value substituted in $yy = xx - xy - 3$ as being the most simple, there results $\frac{4x^6 - 12x^4 + 9xx}{9x^4 - 18xx + 9} = xx - \frac{2x^4 + 3xx}{3xx - 3} - 3$, whence $4x^6 - 12x^4 + 9xx = 9x^6 - 18x^4 + 9x^2 - 6x^6 + 9x^4 + 6x^4 - 9x^2 - 27x^4 + 54x^2 - 27$, which when reduced becomes $x^6 + 18x^4 - 45x^2 + 27 = 0$.

LIII.

If there were more than two unknown quantities and equations given the they may be reduced to

If there were three Equations given and as many unknown Quantities of different Dimensions, mixed promiscuously in each Equation, they may likewise by the foregoing Method be reduced into one, affected with only one unknown Quantity; for neglecting at first one of the three unknown Quantities, two of the three Equations will be sufficient to find an Equation involving the unknown Quantity which was not attended to,

one by the foregoing method.

and either of the two others, and the ſame Operation being performed on one of the Equations employed in the firſt and on the third Equation, there will reſult another Equation involving the ſame unknown Quantities, hence the Queſtion is reduced to that where two Equations involving each two unknown Quantities are concerned, whence a new Equation will be deduced, affected with only one unknown Quantity.

If there were four Equations given and as many unknown Quantities mixed promiſcuouſly in each Equation, the Queſtion will be reduced in like Manner to three Equations and three unknown Quantities, and afterwards to two Equations and two unknown Quantities, and at length to one Equation and one unknown Quantity; and the ſame Method is to be purſued if there were a greater Number of Equations and unknown Quantities.

LIV.

Inconvenience to which the foregoing method is liable.

When the Quantity to be exterminated is of ſeveral Dimenſions, there would be required a very laborious Calculus to exterminate it out of the Equations by the foregoing Method, we ſhall therefore proceed to explain how the Analyſis have remedied this Inconveniency.

Let $\begin{cases} Ax^m + Bx^{m-1} + Cx^{m-2} + Dx^{m-3} + Ex^{m-4} \ldots V = 0 (L) \\ Ax^{m'} + B'x^{m'-1} + C'x^{m'-2} + D'x^{m'-3} + E'x^{m'-4} \ldots V' = 0 (L') \end{cases}$

be two Equations in which $A$, $B$, $C$, $D$, &c. denote any Function of $y$, to exterminate $x$ out of thoſe two Equations, it is ſufficient to find a Function of $x$ by which the firſt Equation being multiplied, and another Function of $x$ by which the ſecond Equation being multiplied in the Sum of the Products, the Terms including the Powers of $x$ will be deſtroyed.

How it has been removed.

let then $\begin{cases} Mx^n + Nx^{n-1} + Px^{n-2} + Qx^{n-3} + Rx^{n-4} \ldots T \\ M'x^{n'} + N'x^{n'-1} + P'x^{n'-2} + Q'x^{n'-3} + R'x^{n'-4} \ldots T' \end{cases}$

be thoſe two Functions. After the Multiplication and Addition of the Products there will reſult,

$$\begin{aligned}
&AMx^{m+n} + BMx^{m+n-1} + CMx^{m+n-2} + DMx^{m+n-3} \ldots VT = 0 \\
&A'M'x^{m'+n'} + ANx^{m+n'-1} + BNx^{m+n'-2} + CNx^{m+n'-3} \ldots V'T' = 0 \\
&\quad + B'M'x^{m'+n'-1} + APx^{m+n'-2} + BPx^{m+n'-3} \\
&\quad + A'N'x^{m'+n'-1} + CM'x^{m'+n'-2} + AQx^{m+n'-3} \\
&\qquad\quad + B'N'x^{m'+n'-2} + DM'x^{m'+n'-3} \\
&\qquad\quad + A'P'x^{m'+n'-2} + CN'x^{m'+n'-3} \\
&\qquad\qquad\quad + BP'x^{m'+n'-3} \\
&\qquad\qquad\quad + AQ'x^{m'+n'-3}
\end{aligned}$$

Where I obſerve, 1°. That as the Condition that the Terms including the Powers of $x$ ſhould be deſtroyed, furniſh the Equations by Means of which the unknown Coeficients $M$, $N$, &c. are to be determined, $m + n$ muſt be equal to $m' + n'$; whence reſults the following Equations for determining thoſe Coeficients.

$$A M + A' M' = 0$$
$$A N + A' N' + B M + B' M' = 0$$
$$A P + A' P' + B N + B' N' + C M + C' M' = 0$$
$$A Q + A' Q' + B P + B' P' + C N + C' N' + D M + D' M' = 0$$
$$A R + A' R' + B Q + B' Q' + C P + C' P' + D N + D' N' + E M + E' M' = 0$$

Consequently, $V T + V' T' = 0$

I obſerve, 2°. That as the Number of Terms including the Powers of $x$ is $m + n$, the Number of Equations they furniſh when ſuppoſed equal to Nothing, will be alſo $m + n$, and as each Term of theſe Equations include an unknown Quantity, Viz. the Coeficient of a Term of the aſſumed Multinomials, it follows (LXXXIX. Chap. 1.) that the Number of undetermined Coeficients $n + 1 + n' + 1$ will be $= m + n + 1$, or $m' + n' + 1$; conſequently $n = m' - 1$ and $n' = m - 1$. Hence when two Equations each involving two unknown quantities are given, to exterminate one of them, it is ſufficient to multiply the firſt Equation by a Multinomial as $M x^{m'-1} + N x^{m'-2} +$, &c. $m'$ denoting the Degree of the ſecond Equation; and the ſecond by a Multinomial as $M' x^{m-1} + N' x^{m-2}$ &c. $m$ denoting the Degree of the firſt Equation: To add thoſe two Products, and to put the Coeficients of each Power of $x$ in this Sum equal to Nothing, the laſt Term of the Sum will be the Equation in $y$.

LV.

For Example, let it be propoſed to exterminate $x$ out of the Equations $A x^2 + B x + C = 0$ and $A' x^2 + B' x + C' = 0$, multiplying $A x^2 + B x + C$ by $M x + N$, and $A' x^2 + B' x + C' = 0$ by $M' x + N'$, and adding the Products we will have $(A M + A' M') x^3 + (B M + A N + B' M' + A' N') x^2 + (C M + B N + C M' + B' N') x + N C + C' N' = 0$, the Condition that the Terms in this Equation including the Powers of $x$ ſhould be deſtroyed will give $A M + A' M' = 0$, $B M + A N + B' M' + A' N' = 0$, $C M + B N + C M' + B' N' = 0$, conſequently $N C + C' N' = 0$, and as there is one Coeficient undetermined. I determine $M$ by ſuppoſing it $= A'$, which gives $M' = - A$, $A N + A' N' = A B' - B A'$, $B N + B' N' = A C - C A'$, from which I deduce $N = \frac{A B' B' - B A B' - A A' C + C A' A'}{A B' - B A'}$, and $N' = \frac{A A C' - A C A' - A B B' + B B A'}{A B' - B A'}$ and thoſe Values of $N$ and $N'$ being ſubſtituted in the Equation $C N + C' N' = 0$ there reſults $A C B' B' - B C A' B' - A C A' C' + C C A' A' + A A C' C' - A C A' C' - A B B' C' + B B A' C' = 0$, that is

Inveſtigation of Sir Iſaac Newton's rule for exterminating an unknown quantity of two dimenſions in each equation by the foregoing Method.

RULE I.

$$(A C' - B B' - 2 C A') A C' + (B C' - C B') B A' + (A B' B' + C A' A') C = 0.$$

If it was propoſed to exterminate $x$ out of the Equations $x x + 5 x - 3 y y = 0$ and $3 x x - 2 x y + 4 = 0$, I reſpectively ſub-

Example.

ſtitute in the preceding Expreſſion for $A, B, C$; $A', B' C'$; theſe Quantities, $1, 5, -3yy; 3, -2y, 4$; and duly obſerving the Signs + and −, there ariſes $(4 + 10y + 18yy) \times 4 + (20 - by^3) \times 15 + (4yy - 27yy) \times -3yy = 0$, or $16 + 40y + 72yy + 300 - 90y^3 + 69y^4 = 0$.

LVI.

Inveſtigation of Sir Iſaac Newton's Rule for exterminating an unknown quantity of three dimenſions in one, and of two dimenſions in the other equation by the foregoing Method.

Let it be propoſed to exterminate $x$ out of the Equations $Ax^3 + Bx^2 + Cx + D = 0$, and $A'x^2 + B'x + C' = 0$, multiplying $Ax^3 + Bx^2 + Cx + D$ by $Mx + N$ and $A'x^2 + B'x + C'$ by $M'x^2 + N'x + P'$ and adding their Products there will reſult $(AM + A'M')x^4 + (BM + AN + B'M' + A'N')x^3 + (CM + BN + CM' + B'N' + A'P')x^2 + (DM + CN + C'N' + B'P')x + DN + C'P' = 0$. The Condition that the Terms of this Equation including the Powers of $x$ ſhould be deſtroyed, will give $AM + A'M' = 0$, $BM + AN + B'M' + A'N' = 0$, $CM + BN + CM' + B'N' + A'P' = 0$, $DM + CN + CN' + B'P' = 0$, conſequently $DN + C'P' = 0$, and as one of the Coeficients is undetermined I determine $M$ by ſuppoſing it $= A'$, which gives $M' = -A$, $AN + A'N' = AB' - BA'$, $BN + B'N' + A'P' = AC' - CA'$, $CN + CN' + B'P' = -DA'$, from thoſe three laſt Equations I deduce the Values of $N$ and $P'$, which may readily be obtained by Means of the Formulas of Article LXXXI. Chap. I. by putting $x = N$, $y = P'$, $z = N'$, $m = AB' - BA'$, $n = AC' - CA'$, $p = -DA'$, $a = A, b = 0, c = A'$; $d = B, f = B', e = A'$; $g = C, b = B', k = C'$, then $x = \frac{ekm - fbm + cbn - cep}{aek - afb + cdb - ceg}$ gives

$$N = \frac{AA'B'C' - BA'A'C' - AB'^3 + BA'B'^2 + AA'B'C - CA'A'B' + DA'^3}{AA'C' - AB'B' + BA'B' - CA'A'}$$

and $y = \frac{akn - gcn + dcp - afp + gfm - dkm}{ack - abf + dbc - ceg}$ becomes

$$N' = \frac{A^2C'^2 - 2ACA'C' + C^2A'^2 - BDA'^2 + ADA'B' + ACB'^2 - BCA'B' - ABB'C' + BBA'C'}{AA'C' - AB'B' + BA'B' - CA'A'}$$

ſubſtituting thoſe Values of $N$ and $N'$ in the Equation $DN + C'P' = 0$, there reſults $ADA'B'C - BDA'A'C - ADB'^3 + BDA'B'B' + ADA'B'C - CDA'A'B' + DDA'^3 + AAC'^3 - ACA'C'C' - ACA'C'C' + CCA'A'C' - BDA'A'C + ADA'B'C' + ACB'B'C' - BCA'B'C' - ABB'C'C + BBA'C'C' = 0$.

Or $AAC'^3 - ABB'C'^2 - 2ACA'C'C' + B^2A'C'C' - BCA'B'C' - 2BDA'A'C' + ACB'B'C' + CCA'A'C - ADB'^3 - CDA'^2B' + 3ADA'B'C' + BDA'B'^2 + DDA'^3 = 0$, that is,

RULE II.

$(AC' - BB' - 2CA')AC'C' + (BC' - CB' - 2DA')BA'C' + (CC' - DB')(AB'B' + CA'A') + (3AB'C' + BB'B' + DA'A')DA' = 0$.

Example. If it was propoſed to exterminate $x$ out of the Equations $y^3 - xyy - 3x = 0$ and $yy + xy - xx + 3 = 0$, I ſubſtitute in the precedent Expreſſion

for $A, B, C, D$; $A', B' C'$, and $x$, thoſe Quantities $1, -x, 0, -3x$; $1, x, -xx+3$, and $y$ reſpectively; and there comes out $(3-xx+xx)\times(9-6xx+x^4)+(-3x+x^3+6x)(-3x+x^3)+3xx\times xx+(9x-3x^3-x^3-3x)\times-3x=0$, then blotting out the ſuperfluous Quantities and multiplying, you have $27-18xx+3x^4-9xx+x^6+3x^4-18x^2+12x^4=0$, and ordering $x^6+18x^4-45xx+27=0$.

LVII.

Inveſtigation of Sir Iſaac Newton's Rule for exterminating an unknown quantity of four dimenſions in one, and of two dimenſions in the other equation by the foregoing Method.

Let it be propoſed to exterminate $x$ out of the Equations $Ax^4+Bx^3+Cx^2+Dx+E=0$, and $A'x^2+B'x+C=0$; I multiply the firſt by $Mx+N$ and the ſecond by $M'x^3+N'x^2+O'x+Q'$, and adding their Products there reſults, $(AM+A'M')x^5+(BM+AN+B'M'+A'N')x^4+(CM+BN+CM'+B'N'+A'O')x^3+(DM+CN+CN'+B'O'+A'Q')x^2+(EM+DN+B'Q'+CO')x+EN+CQ'=0$, and the Condition that all the Terms in this Equation including the Powers of $x$ ſhould be deſtroyed, gives the Equations $AM+A'M'=0$, $BM+AN+B'M'+A'N'=0$, $CM+BN+CM'+B'N'+A'O'=0$, $DM+CN+CN'+B'O+A'Q'=0$, $EM+DN+B'Q'+C'O'=0$, conſequently $EN+C'Q'=0$, and as one of the Coeficients is undetermined I determine $M$ by ſuppoſing it equal to $A'$; whence

$M'=-A$, $AN+A'N'=AB'-A'B$, $BN+B'N'+A'O'=AC'-CA'$, $CN+A'Q'+B'O'+CN'=-DA'$, $DN+B'Q'+C'O'=-EA'$.

I deduce from thoſe four laſt Equations the Values of $N$ and $Q'$ by Help of the Formulas of Art. LXXXVII, by putting $x=N, y=N', z=O', t=Q', a=A, b=A', c=0, d=0$, $a=AB'-BA'$, $d'=B$, $b'=B'$, $c'=A'$, $d'=0$, $a'=AC'-A'C$, $a''=C$, $b''=C'$, $c''=B'$, $d'''=A'$, $a''=-A'D$, $a'''=D$, $b'''=0$, $c'''=C'$, $b''=C$, $c''=B'$, $d''=A'$, $A''=-A'D$, $c'''=D$, $b'''=0$, $c'''=C'$, $d'''=B'$, $a^{IV}=-A'E$.

In this Caſe the Numerator of the Value of $x$ is reduced to $[(ab'-a'b)c''+(a''b-ab'')c']d'''+[(a'b-ab')c'''-a'''bc']d'''$

Conſequently the Numerator of the Value of $N$ will be $AB'^4-BA'B'^3-AA'B'^2C'+CA'^2B'^2-DA'^3B'-AA'B'^2C'+BA'^2B'C'+AA'^2C'^2-CA'^3C'-AA'B'^2C+EA'^4+BA'^2B'C'$.

And the Numerator of the Value of $t$ being reduced to $[(ab'-a'b)c''+(a''b-ab'')c']a'''+[(a'b-ab')c'''-a'''bc']a''+[a'''bc''+(ab''-a''b)c''']a'+[-a'''b'c'+a'''b''c'+(a''b'-a'b'')c''']a$.

Conſequently, the Numerator of the Value of $Q'$ will be $-AEA'B'^2+BEA'^2B'-CEA'^3+AEA'^2C'-DBA'^2C'+ADA'B'C'+D^2A'^3+ADA'B'C-CDA'^2B'+A^2C'^3-2ACA'C'^2+C^2A'^2C'-ADB'^3+BDA'B'^2+ADA'B'C'-BDA'^2C'+ACB'^2C'-BCA'B'C'-ABB'C'^2+B^2A'C'^2$

wherefore $E N + C Q' = 0$ will be transformed into
$AEB'^4 - BEAB'^3 - AEAB'^2C + CEA^2B'^2 - DEA^3B' - AEAB'^2C'$
$+ BEA^2B'C' + AEA^2C^2 - CEA^3C' - AEAB'^2C + BEA^2B'C' + E^2A^4$
$- AEAB'^2C + BEA^2B'C - CEA^3C' + AEA^2C'^2 - BDA^2C'^2$
$+ ADAB'C'^2 + D^2A^3C' + ADAB'C'^2 - CDA^2B'C'$
$+ A^2C'^4 - 2ACA'C'^3 + C^2A'^2C'^2 - ADB'^3C' + BDAB'^2C'$
$+ ADAB'C'^2 - BDA^2C'^2 + ACB'^2C'^2 - BCAB'C'^2 - ABB'C^3$
$+ B^2A'C^3 = 0$. Or into,
$A^2C^4 - 2ACA'C'^3 - ABB'C^3 + B^2A'C^3 - BCAB'C'^2 - 2BDA^2C'^2$
$+ ACB'^2C^2 - ADB'^3C + AEB'^4 - 2AEAB'^2C + C^2A^2C^2$
$- CDA^2B'C' + CEA^2B'^2 - 2CEA^3C + 3ADAB'C'^2$
$+ BDAB'^2C + D^2A^3C' + 2AEA^2C'^2 + 3BEA^2B'C' - DEA^3B'$
$+ E^2A^4 - BEAB'^3 - 2AEAB'^2C = 0$.

RULE III.

That is $(AC' - BB' - 2CA')AC'^3 + (BC - CB' - 2DA')BAC'^2$
$+ (AB'^2 + CA'^2)(CC'^2 - DB'C + EB'B' - 2EAC')$
$+ (3AB'C + BB'^2 + DA'^2)DA'C + (2AC'^2 + 3BB'C - DAB'$
$+ EA'^2)EA'^2 + (-BB' - 2AC')EAB'^2 = 0$.

If it was propoſed to exterminate $x$ out of the Equations
$x^4 + yyx^2 - 140xx + y^4 = 0$, and $xx + yx - 20x + yy = 0$;

Example. Subſtitute for $A$, $B$, $C$, $D$, $E$; $A'$, $B'$, $C'$, reſpectively, $1$, $0$, $yy - 140$, $0$, $y^4$; $1$, $y - 20$, and $yy$; and there will come out
$(-yy + 280) \times y^5 + (2yy - 40y + 260) \times (260y^4 - 40y^5)$
$+ 3y^4 \times y^4 - 2yy \times (y^6 - 40y^5 + 400y^4) = 0$; and by Multiplication $1600y^6 - 20800y^5 - 67600y^4 = 0$, and by Reduction
$4yy - 52y + 169 = 0$.

LVIII.

Inveſtigation of Sir Iſaac Newton's rule for exterminating an unknown quantity of three dimenſions in each equation by the foregoing Method.

Let it be propoſed to exterminate $x$ out of the Equations
$Ax^3 + Bx^2 + Cx + D = 0$, $A'x^3 + B'x^2 + C'x + D' = 0$, multiplying $Ax^3 + Bx^2 + Cx + D$ by $Mx^2 + Nx + O$, and $A'x^3 + B'x^2 + C'x + D'$ by $M'x^2 + N'x + O'$, and adding their Products there will reſult $(AM + A'M')x^5 + (BM + AN + B'M' + A'N')x^4$
$+ (CM + BN + AO + C'M' + B'N' + A'O')x^3 + (DM + CN$
$+ BO + D'M' + C'N' + B'O')x^2 + (DN + CO + D'N' + C'O')x$
$+ OD + O'D' = 0$.

The Condition that the Powers of $x$ ſhould be deſtroyed in this Equation will give $AM + A'M' = 0$, $BM + AN + B'M' + A'N' = 0$
$CM + BN + AO + C'M' + B'N' + A'O' = 0$, $DM + CN + BO$
$+ D'M' + C'N' + B'O' = 0$, $DN + D'N' + CO + C'O' = 0$,
conſequently $DO + D'O' = 0$, and as there is one of the Coeficients indetermined, I determine $M$ by ſuppoſing it $= A'$; whence
$M' = -A$, $AN + A'N' = AB' - BA'$, $BN + B'N' + AO + A'O'$
$= AC' - CA'$, $CN + C'N' + BO + B'O' = AD' - DA'$,
$DN + D'N' + CO + C'O' = 0$.

I deduce from the four laſt Equations the Values of $O$ and $O'$ by the Help of the Formulas of Art. LXXXVII. by putting
$x = N$, $y = N'$, $z = O$, $t = O'$, $a = A$, $b = A'$, $c = o$, $d = o$, $a = AB' - BA'$; $a' = B$, $b' = B'$, $c' = A$, $d' = A$, $a' = AC - CA'$; $a'' = C$, $b'' = C'$, $c'' = B$, $d'' = B'$, $a'' = AD' - DA'$; $a''' = D$, $b''' = D'$, $c''' = C$, $d''' = C$, $a''' = o$.

In this Caſe the Numerator of the Value of $z$ will be reduced to
$[(ab' - a'b)a'' + (a'b - ab'')a' + (a'b'' - a''b')a]d''' + [(ab''' - a'''b)a' + (a'''b' - a'b''')a]d'' + [(a'''b - ab''')a'' + (a''b''' - a'''b'')a]d'$
conſequently the Numerator of the Value of $O$ will be,
$A^2B'CD' - ABACD' - ADAB'C + BDA^2C + ACAC' - A^2C^3 - C^2A^2C' + ACAC'^2 + ABB'C^2 - ACB'^2C - B^2AC'^2 + BCAB'C + A^2B'C'D' - ADAB'C' - ACAB'D' + CDA^2B' + ADB'^3 - ABB'^2D' - BDAB'^2 + B^2AB'D' + ADA^2D' - A^2AD'^2 - D^2A^3 + ADA^2D' + ACAB'D' - ADAB'C - BCA^2D' + BDA^2C'$.

And the Numerator of the Value of $t$ being reduced to
$[(a'b - ab')c''' + (ab''' - a'''b)c']a'' + [(a'''b - ab''')c'' + (ab'' - a''b)c''']a' + [(a'b''' - a'''b')c'' + (a'''b'' - a''b''')c' + a''b' - a'b'')c''']a$.

Conſequently the Numerator of the Value of $O'$ will be
$ABCA'D' - A^2CB'D' - BCDA'^2 + ACDA'B' + A^2D'^2 - A^2DA'D' - A^2DA'D' + AD^2A^2 + ABDA'C' - A^2BC'D' - BDCA^2 + ABCA'D' + A^2CC'^2 - AC^2AC - AC^2AC + C^3A^2 + AB^2B'D' - ABDB'^2 - B^3A'D' + B^2DA'B' + A^2DB'C' - A^2CB'D' - ABDA'C + ABCA'D' + AC^2B'^2 - ABCB'C' - BC^2A'B' + B^2CA'C'$.

Wherefore $DO + D'O' = o$, will be transformed into
$A^2DB'CD' - ABDA'CD' - AD^2A'B'C' + BD^2A'^2C + ACDA'C'^2 - A^2DC'^3 - C^2DA'^2C + ACDA'C'^2 + ABDB'C'^2 - ACDB'^2C' - B^2DA'C'^2 + BCDA'B'C' + A^2DB'C'D' - AD^2A'B'C' - ACDA'B'D' + CD^2A'^2B' + AD^2B'^3 - ABDB'^2D' - BD^2A'B'^2 + B^2DA'B'D' + AD^2A'^2D' - A^2DA'D'^2 - D^3A'^3 + AD^2A'^2D' + ACDA'B'D' - AD^2A'B'C' - BCDA'^2D' + BD^2A'^2C' + ABCA'D'^2 - A^2CB'D'^2 - BCDA^2D' + ACDA'B'D' + A^3D'^3 - A^2DA'D'^2 - A^2DA'D'^2 + AD^2A^2D' + ABDA'C'D' - A^2BCD'^2 - BCDA'^2D' + ABCAD'^2 + A^2CC'^2D' - AC^2A'C'D' - AC^2A'C'D' + C^3A^2D' + AB^2B'D'^2 - ABDB'^2D' - B^3A'D'^2 + B^2DA'B'D' + A^2DB'C'D' - A^2CB'D'^2 - ABDA'C'D' + ABCA'D'^2 + AC^2B'^2D' - ABCB'CD' - BC^2A'B'D' + B^2CA'C'D' = o$, or into
$-A^2DC'^3 + ABDB'C'^2 + 2ACDA'C'^2 + A^2CC'^2D' - ABCB'CD' - 2AC^2A'CD' - ABDA'C'D' - B^2DA'C'^2 + BCDA'B'C'$

$+2BD^2A^2C' + A^3D'^3 - A^2BCD'^2 - 2A^2CB'D'^2 - 3A^2DAD'^2$
$- ACDB'^2C' + AD^2B^3 + AC^2B'^2D' - 2ABDB'^2D' - C^2DA'^2C'$
$+ CD^2A'^2B' + C^3A'^2D' - 2BCDA'^2D' - 3AD^3A'B'C' - BD^2A'B'^2$
$- D^3A'^3 + 3AD^2A'^2D' + 3ABCA'D'^2 + B^2CA'C'D' - BC^2A'B'D'$
$- BCDA'^2D' - B^3A'D'^2 + 2B^2DA'B'D' + AB^2B'D'^2 + 3A^2DB'C'D'$
$+ ACDA'B'D' = 0.$ That is,

RULE IV.

$(AC' - BB' - 2CA') \times (ADC'^2 - ACC'D') + (AD' + BC'$
$- CB' - 2DA')BDA'C' + (-AD' + BC' - 2CB' + 3DA') \times AAD'^2$
$+ (CDC' - DDB' - CCD' + 2BDD') \times (AB'^2 + CA'^2)$
$+ (3AB'C' + BB'B' + DA^2 - 3AA'D')DDA' + (-3AD'$
$- BC' + CB' + DA')BCA'D' + (BD' - 2DB') \times BBA'D'$
$+ (-BBD' - 3ADC' - CDA') \times AB'D' = 0.$

LIX.

Let the Equations $Ax^4 + Bx^3 + Cx^2 + Dx + E = 0$, and $A'x^4 + B'x^3 + C'x^2 + D'x + E' = 0$, be propoſed; the reſulting Equation after $x$ is exterminated will be found to be,

RULE V.

$$\begin{aligned}
&(EA' - AE')^4 - 3(BA' - AB')(EB' - BE') \\
&\quad - 2(CA' - AC')(EC' - CE') \\
&\quad - (DA' - AD')(EB' - BE') \\
&\quad - (DA' - AD')(DC' - CD') \\
&\qquad \times \\
&(A'E - AE')^2
\end{aligned}$$

$$\begin{aligned}
&+ 3(BA' - AB')(EB' - BE')(EC' - CE') \\
&- 3(BA' - AB')(DB' - BD')(ED' - DE') \\
&+ 2(DA' - AD')^2(EC' - CE') \\
&+ (CA' - AC')(EB' - BE')^2 \\
&+ 2(CA' - AC')(DA' - AD')(ED' - DE') \\
&- (CA' - AC')(CB' - BC')(ED' - DE') \\
&\qquad \times \\
&(EA' - AE')
\end{aligned}$$

$$\begin{aligned}
&+ 2(BA' - AB')(DA' - AD')(EB' - BE') \\
&- (DA' - AD')^3 - (BA' - AB')(DB' - BD')^2 \\
&+ 2(BA' - AB')(DA' - AD')(DC' - CD') \\
&- (DC' - CD')(CA' - AC')^2 \\
&+ (EB' - BE')(CA' - AC')^2 \\
&+ (CA' - AC')(DA' - AD')(DB' - BD') \\
&+ (BA' - AB')(CB' - BC')(EB' - BE') \\
&+ (BA' - AB')(CB' - BC')(DC' - CD') \\
&\qquad \times \\
&(ED' - DE')
\end{aligned}$$

$$\begin{aligned}
&+ (CA' - AC')^2(EC' - CE') \\
&- 2(BA' - AB')(DA' - AD')(EC' - CE') \\
&- (BA' - AB')(CB' - BC')(EC' - CE') \\
&+ 2(BA' - AB')(EB' - BE')(DB' - BD') \\
&- (CA' - AC')(DA' - AD')(EB' - BE') \\
&\qquad \times \\
&(EC' - CE') \\
&- (BA' - AB')(EB' - BE') \\
&- (BA' - AB')(DC' - CD') \\
&\qquad \times \\
&(EB' - BE')^2.
\end{aligned}$$

This Rule will ſerve to exterminate an unknown Quantity of a Number of Dimenſions in each Equation not exceeding four; for if the unknown Quantity is only of three Dimenſions in the Equation $L'$, we have no more to do than to put $A' = 0$, that is, to omit all the Terms in the reſulting Equation among whoſe Factors $A'$ is included, and then the whole Equation will be diviſible by $A$. And if the unknown Quantity is only of two Dimenſions in the Equation $L'$, we are to omit alſo all the Terms among whoſe Factors $B'$ is included; and the reſulting Equation will be again diviſible by $A$, &c. But if the unknown Quantity riſes only to three Dimenſions in the Equation $L$, we are to omit all the Terms including $E$, and the whole Equation will be diviſible by $E'$. And if the unknown Quantity riſes only to two Dimenſions in the Equation $L$, we are alſo to omit all the Terms which include $D$, and to divide a ſecond Time the Equation by $E'$, and ſo on.

How an unknown quantity of a number of dimenſions not exceeding four in each equation may be exterminated by the preceding Rule.

For Example, to exterminate $x$ out of thoſe two Equations $x^3 - 2ax^2 + 4ayx - y^3 = 0$, and $ax^2 + y^2x - ay^2 = 0$, by the foregoing Rule; I put $E' = -ay^2$, $D' = y^2$, $C' = a$, $B' = 0$, $A' = 0$; $A = 1$, $B = -2a$, $C = 4ay$, $D = -y^3$, and $E = 0$.

Example.

Whence the Terms in the Formula, among whoſe Factors $A'$, $B'$, or $E$ are included, will vaniſh, and the remaining Terms will be diviſible by $E'$ and $A^2$; conſequently the Formula will be reduced to

$$A^2E'^3 - 2ACC'E'^2 - ABD'E'^2 + 3ADD'C'E' + ACD'^2E' + B^2C'E'^2 - 2BDC'^2E' - ADD'^3 + D^2C'^3 - CDC'^2D' + BDC'D'^2 + C^2C'^2E' - BCC'D'E' = 0.$$

In which ſubſtituting for $E'$, $D'$, $C'$; $A$, $B$, $C$, $D$, their Values,

$$-a^3y^6 - 8a^4y^5 + 2a^3y^6 + 3a^2y^7 - 4a^2y^7 + 4a^5y^4 + 4a^4y^5 + y^9 + a^3y^6 + 4a^3y^6 + 2a^2y^7 - 16a^5y^4 - 8a^4y^5$$

which after the Terms are reduced and ordered becomes,

$$y^9 + a^2y^7 + 6a^3y^6 - 12a^4y^5 - 12a^5y^4 = 0 = y^5 + a^2y^3 + 6a^3y^2 - 12a^4y - 12a^5.$$

LX.

The foregoing Method for exterminating an unknown Quantity of ſeveral Dimenſions in two Equations, each involving two unknown Quantities, alſo ſerves to determine the Degree of the reſulting Equation.

How the degree of the reſulting equation is determined.

Let the higheſt Dimenſions of $y$ in $A$ be $p$, of $y$ in $B$, $p + 1$, of $y$ in $C$, $p + 2$, &c. In like Manner let the higheſt Dimenſion of $y$ in $A'$ be $p'$, of $y$ in $B'$ be $p' + 1$, of $y$ in $C'$, $p' + 2$, &c. Then the higheſt Dimenſions of $y$ in the Coeficients of each of the unknown Quantities, included in the Equations ariſing from the Suppoſition, that the Terms affected by the Powers of $x$ in the Sum of the two Products of the propoſed Equations into the undetermined Multinomials ſhould be deſtroyed, form ſo many arithmetical Progreſſions, (it is thus, a Series of Quantities, that increaſe or decreaſe by the ſame conſtant Difference, is called) having the ſame common Difference.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| * | * | * | * | * | * | * | * | $p$ | $p'$ |
| * | * | * | * | * | * | $p$ | $p'$ | $p+1$ | $p'+1$ |
| * | * | * | * | $p$ | $p'$ | $p+1$ | $p'+1$ | $p+2$ | $p'+2$ |
| * | * | $p$ | $p'$ | $p+1$ | $p'+1$ | $p+2$ | $p'+2$ | $p+3$ | $p'+3$ |
| $p$ | $p'$ | $p+1$ | $p'+1$ | $p+2$ | $p'+2$ | $p+3$ | $p'+3$ | $p+4$ | $p'+4$ |
| &c. | &c. | &c. | &c. | &c. | &c. | &c. | &c. | &c. | &c. |

As will appear by the Inſpection of thoſe Equations, (Art. LIV.)

It will alſo appear by the Inſpection of the Formulas of Art. LXXXVII. Chap. I. that each of the Terms of the reſulting Equation after the unknown Quantities are exterminated, conſiſts of the ſame Number $m+n+1$ of the Coeficients of thoſe unknown Quantities, and that two Coeficients of the ſame unknown Quantity do not enter the ſame Term: Whence it is eaſy to conclude, that the higheſt Dimenſion of $y$ in the reſulting Equation will be expreſſed by the Sum of $m+n+1$ Terms of $m+n+1$ the foregoing Progreſſions, of which Terms two are not in the ſame Column or upon the ſame Line. Now upon Examination it will be found that $m+n+1$ Terms of $m+n+1$ of thoſe Progreſſions taken in this Manner will always make the ſame Sum. For Example, If we take five Terms of five of thoſe Progreſſions in ſuch a Manner that two of thoſe Terms are not contained in the ſame Column, or are upon the ſame Line, their Sum will be found to be $3p'+2p+6$; and the general Expreſſion of this Sum will be $S+(m+n+1)\times\left(\frac{m+n}{2}\right)$, $S$ denoting the Sum of the Terms which compoſe the firſt Line, if the Progreſſions were continued ſo far; that is, of $p$, $p-1$, $p-2$, $p-3$, continued to $n+1$ Terms, and of $p'$, $p'-1$, $p'-2$, $p'-3$, &c. continued to $n'+1$ Terms; wherefore $S=(2p-n)\frac{n+1}{2}+(2p'-n')\times\left(\frac{n'+1}{2}\right)$; conſequently, if the higheſt Dimenſion of $y$ in the reſulting Equation be expreſſed by $G$, then

$$G=(m+n+1)\left(\frac{m+n}{2}\right)+(2p+n)\frac{n+1}{2}+(2p'+n')\frac{n'+1}{2};$$

in which ſubſtituting for $n$ and $n'$ their Values $m'-1$ and $m-1$ there reſults $G=mm'+pm'+p'm$.

LXI.

Limit which the degree of the reſulting equation can not exceed.

It is eaſy to perceive, that the Degree of the reſulting Equation when an unknown Quantity of ſeveral Dimenſions in each Equation is exterminated, can never exceed $mm'+pm+p'm$, $p$ and $p'$ expreſſing the Exceſs of the greateſt Sum of the Exponents of the unknown Quantities in any one Term above the greateſt Exponent of the Quantity to be exterminated in each Equation; for Example, if there were given the two Equations $a^3x^5y-2a^4y^2x^3+y^3x-a^9=0$, and

$a^3x^3 - 3a^3xy^2 + y^5x - y^6 - a^4 = 0$; the greateſt Sum of the Exponents of $x$ and $y$ in any one Term of the firſt Equation is 9, which ſubtracted from 5 the greateſt Exponent of $x$ gives $p = 4$; and the greateſt Sum of the Exponents of $x$ and $y$ in any one Term of the ſecond Equation which is 6 ſubtracted from 3 the greateſt Exponent of $x$ gives $p' = 3$, now $x$ being exterminated out of the Equations $ay^4x^5 - 2a^4y^6x^3 + y^8x - a^9 = 0$, $a^3y^3x^3 - 3a^3y^5x + y^5x - y^6 - a^4 = 0$, the Degree of the reſulting Equation by the foregoing Art. will not exceed $5 \times 3 + 4 \times 3 + 5 \times 3$, that is, the 42d. Degree, much leſs then will the Equation in $y$ deduced from the two propoſed Equations aſcend to the 42d. Degree.

LXII.

Let $\begin{cases} Ax^m + Bx^{m-1} + Cx^{m-2} + Dx^{m-3} + \dots V = 0 \\ A'x^{m'} + B'x^{m'-1} + C'x^{m'-2} + D'x^{m'-3} + \dots V' = 0 \\ A''x^{m''} + B''x^{m''-1} + C''x^{m''-2} + D''x^{m''-3} + \dots V'' = 0 \end{cases}$

be three Equations; in which $A$, $B$, $C$, &c. $A'$, $B'$, $C'$, &c. $A''$, $B''$, $C''$, &c. denote Functions of two unknown Quantities $y$ and $z$, and known Quantities, and let the Dimenſions of $A$, $B$, $C$, &c. be expreſſed by $p$, $p+1$, $p+2$, &c. thoſe of $A'$, $B'$, $C'$, &c. by $p'$, $p'+1$, $p'+2$, &c. and thoſe of $A''$, $B''$, $C''$, &c. by $p''$, $p''+1$, $p''+2$, &c. to obtain the two Equations which reſult when $x$ is exterminated, it ſuffices to find three Functions of $x$ as ſimple as poſſible, by which the three Equations being multiplied reſpectively, in the Sum of the three Products the Terms affected by the Powers of $x$ will be deſtroyed, and three other Functions of $x$ different from the firſt, as ſimple alſo as poſſible endued with the ſame Property.

How an unknown quantity of ſeveral dimenſions is exterminated out of three equations by the foregoing method.

Let $\begin{cases} Mx^n + Nx^{n-1} + Px^{n-2} + Qx^{n-3} + \dots T \\ M'x^{n'} + N'x^{n'-1} + P'x^{n'-2} + Q'x^{n'-3} + \dots T' \\ M''x^{n''} + N''x^{n''-1} + P''x^{n''-2} + Q''x^{n''-3} + \dots T'' \end{cases}$

be the three firſt Functions, performing the Multiplications, and adding the three Products there will reſult,

$$\begin{aligned}
&AMx^{m+n} + BMx^{m+n-1} + CMx^{m+n-2} + DMx^{m+n-3} \dots VT = 0 \\
&A'M'x^{m'+n'} + AN'x^{m+n'-1} + BN'x^{m+n'-2} + CN'x^{m+n'-3} \dots V'T' \\
&A''M''x^{m''+n''} + B'M''x^{m'+n''-1} + AP'x^{m+n'-2} + BP'x^{m+n'-3} \dots V''T'' \\
&\quad + A'N''x^{m'+n''-1} + CM'x^{m+n'-2} + AQ'x^{m+n'-3} \\
&\quad + B''M''x^{m''+n''-1} + BM'x^{m+n'-2} + DM'x^{m+n'-3} \\
&\quad + A''N''x^{m''+n''-1} + AP'x^{m+n'-2} + CN'x^{m+n'-3} \\
&\quad + CM''x^{m''+n''-2} + BP'x^{m+n'-3} \\
&\quad + BN''x^{m''+n''-2} + AQ'x^{m+n'-3} \\
&\quad + AP''x^{m''+n''-2} + DM''x^{m''+n''-3} \\
&\quad + CN''x^{m''+n''-3} \\
&\quad + BP''x^{m''+n''-3} \\
&\quad + AQ''x^{m''+n''-3}
\end{aligned}$$

LXIII.

Let us ſuppoſe firſt $m+n=m'+n'$ and $m+n>m''+n''$, or at furtheſt equal. It is obvious that in Order to deſtroy all the Terms affected with $x$, the Number of undetermined Coeficients muſt be $m+n+1$, wherefore $m+n+1=n+1+n'+1+n''+1$, wherefore $n'=m-n''-2$ and $n=m'-n''-2$. This being ſuppoſed, putting the Coeficients of each Power of $x$ equal to Nothing, there will reſult a Series of Equations; in which it is eaſy to perceive, that the higheſt Dimenſions of the ſame unknown Quantity, $M$ or $N$ &c. form an arithmetical Progreſſion, and that all thoſe Progreſſions have the ſame common Difference; whence by a ſimilar Reaſoning to that employed in Artticle LX. it will appear, that the Number $G$ expreſſing the higheſt Dimenſion of the Equation in $y$ and $z$ reſulting from the Combination of $m+n+1$ Equations, will be $G=S+(m+n+1)\frac{m+n}{2}$, $S$ expreſſing the Sum of the Terms which compoſe the firſt Line if the Progreſſions were continued ſo far. Now it is eaſy to perceive

1°. That the Numbers which denote the Dimenſions which the Coeficients $M$, $N$, $P$, &c. ſhould have in the firſt Equation if they were found there, are $p$, $p-1$, $p-2$, $p-3$, &c. the Number of Terms being $n+1$, whoſe Sum is expreſſed by $(2p-n)\times\left(\frac{n+1}{2}\right)$.

2°. That the Numbers which in the ſame Suppoſition denote the Dimenſions of $M'$, $N'$, $P'$, &c. are $p'$, $p'-1$, $p'-2$, &c. continued to a Number of Terms $=n'+1$, whoſe Sum is expreſſed by $(2p'-n')\times\left(\frac{n'+1}{2}\right)$. 3°. In Order to find the Numbers which denote in the firſt Equation the Dimenſions of $M''$, $N''$, $P''$, &c. if thoſe Quantities were found there, I obſerve, that as the Place of the Term $A''M''x^{m''+n''}$ in the Order of the Powers of $x$ is denoted by $m+n-m''-n''+1$; ſo will its Place in the Order of the Equations be denoted by the ſame Number; conſequently, to diſcover what ſhould be the Dimenſion of the Coeficient $M''$ in the firſt Equation, we muſt retrograde $m+n-m''-n''$ Equations; conſequently, the Number which denotes this Dimenſion will be $p''-m-n+m''+n''$; and conſequently, the Numbers which denote in the firſt Equation the Dimenſions of $M''$, $N''$, $P''$, &c. if thoſe Quantities were found there, are $p''-m-n+m''+n''$, $p''-m-n+m''+n''-1$, $p''-m-n+m''+n''-2$, &c. continued to a Number of Terms $=n''+1$, whoſe Sum will be $(2p''-2m-2n+2m''+2n'')\times\left(\frac{n''+1}{2}\right)$.

Conſequently, $G=(m+n+1)\left(\frac{m+n}{2}\right)+(2p-n)\left(\frac{n+1}{2}\right)+(2p'-n')\left(\frac{n'+1}{2}\right)+(2p''-2m-2n+2m''+n'')\left(\frac{n''+1}{2}\right)$,

Or by ſubſtituting for $n$ and $n'$ their Values found above, $G = m\,m' + p\,m' + p'\,m - m - m' + m'' - p - p' + p'' + 1 - (p + p' - p'' + m + m' - m'' - n'' - 2)\,n''$. A Quantity which is undetermined ſince $n''$ has not been determined.

LXIV.

Hence the Equation in $y$ and $z$ may be obtained different Ways, but it is manifeſt that that which gives for $G$ the leaſt Quantity is only to be choſen; and it is from this Condition that $n''$ is to be determined. To determine therefore this extreme Value of $G$ we are to find the limitting Ratio of the Cotemporary Increments of the two undetermined Quantities $G$ and $n''$ in the Equation $G = m\,m' + p\,m' + p'\,m - m - m' + m'' - p - p' + p'' + 1 - (p + p' - p'' + m + m' - m'' - n'' - 2)\,n''$, and to put this Ratio equal to Nothing. How $n''$ is determined and conſequently $G$.

Let $d\,G$ and $d\,n''$ denote the cotemporary Increments of $G$ and $n''$, then $G + d\,G = m\,m' + p\,m' + p'\,m - m - m' + m'' - p - p' + p'' + 1 - p\,n'' - p\,d\,n'' - p'\,n'' - p'\,d\,n'' + p''\,n'' + p''\,d\,n'' - m\,n'' - m\,d\,n'' - m'\,n'' - m'\,d\,n'' + m''\,n'' + m''\,d\,n'' + n''\,n'' + 2\,n''\,d\,n'' + d\,n''\,d\,n'' + 2\,n'' + 2\,d\,n''$; and ſince $G = m\,m' + p\,m' + p'\,m - m - m' + m'' - p - p' + p'' + 1 - p\,n'' - p'\,n'' + p''\,n'' - m\,n'' - m'\,n'' + m''\,n'' + n''\,n'' - 2\,n''$. $d\,G = -p\,d\,n'' - p'\,d\,n'' + p''\,d\,n'' - m\,d\,n'' - m'\,d\,n'' + m''\,d\,n'' + 2\,n''\,d\,n'' + d\,n''\,d\,n'' + 2\,d\,n''$.

And $\frac{d\,G}{d\,n''} = -p - p' + p'' - m - m' + m'' + 2\,n'' + d\,n'' + 2$; wherefore, $-p - p' + p'' - m - m' + m'' + 2\,n'' + d\,n'' + 2$, expreſſes the Ratio of any cotemporary Increments of $G$ and $n''$, and this Ratio is always greater than $-p - p' + p'' - m - m' + m'' + 2\,n'' + 2$; but $d\,n''$ decreaſing, this Quantity will alſo decreaſe, and as we may take $d\,n''$ as ſmall as we pleaſe, we may make $-p - p' + p'' - m - m' + m'' + 2\,n'' + d\,n'' + 2$ approach as near as we pleaſe to $-p - p' + p'' - m - m' + m'' + 2\,n'' + 2$; wherefore, $-p - p' + p'' - m - m' + m'' + 2\,n'' + 2$ is the limit of $-p - p' + p'' - m - m' + m'' + 2\,n'' + d\,n'' + 2$; that is, of the Ratio $\frac{d\,G}{d\,n''}$; which gives $n'' = \frac{m + m' + m'' + p + p' + p''}{2} - m'' - p'' - 1$ when made equal to Nothing. But it will not always be poſſible to employ this Value, 1°. Becauſe $n''$ ſhould be a poſitive Quantity. 2°. Becauſe $\frac{m + m' + m'' + p + p' + p''}{2}$ cannot always be reduced to an Integer. 3°. Becauſe $m + n$ ſhould be greater than $m'' + n''$, or at leaſt equal to it. 4°. Becauſe $m'$ being ſuppoſed leſs than $m$ or $= m$, it follows, that $m' - 2$ ſhould be greater than $n''$, or at leaſt $= n''$. We ſhall therefore

assume $n'' = \frac{m+m'+m''+p+p'+p''+a}{2} - m'' - p'' - 1$

supposing, $a$ to be the least Quantity that can answer those Conditions.

This Value of $n''$ being substituted in $G$ there will result,

$G = m\,m' + m\,m'' + m\,p' + m\,p'' + m'\,m'' + m'\,p + m'\,p'' + m''\,p + m''\,p' + p\,p'' + p'\,p'' + \frac{a^2}{4} - \left(\frac{p+p'+p''+m+m'+m''}{2}\right)^2$.

LXV.

Though there cannot be found a Value for $a$ which will answer those four Conditions, we are not to conclude, that from the Combination of the three Equations a more simple Equation cannot be derived, than if they were combined two by two; for we are to observe, that those Conditions arise from the Supposition, that $m+n = m'+n'$, and $m+n > m''+n''$ &c. now there was no Reason why we should suppose $m+n = m'+n'$, rather than $m+n = m''+n''$ or $m'+n' = m''+n''$. It will therefore be proper to make three different Tryals. And if they all three give positive Values for $n$, $n'$, $n''$, (as small a Quantity as possible being put for $a$) we are to employ the two Results which give the least Value for $G$. and if there result positive Values for $n$, $n'$, $n''$, only two different Ways, they are to be employed, provided they give a less Value for $G$ than would result from the Combination of the Equations two by two, and if there can be found positive Values for $n$, $n'$, $n''$, but one Way, we are to conclude, that the unknown Quantity is once to be exterminated by combining only two of the three Equations together. Finally, if there be no Possibility of obtaining for $n$, $n'$, $n''$, positive Values without rendering the Value of $G$ greater than it would be if the Equations were combined two by two, we are to have Recourse to this last Method, but this last Case will scarcely ever happen, if it happens at all: For Example, it can never occur when $p = p' = p'' = 0$, one Equation at least will be found more simple than by combining the Equations two by two, except when some of the Quantities $m$, $m'$, $m''$, are equal to 1.

Observations which serve to find the values of n, n', n''.

LXVI.

The foregoing Principles are sufficient for finding in all Cases the two Equations in $y$ and $z$. We shall now proceed to illustrate them by some Examples.

1°. Let $p = p' = p'' = 0$, and $m = m' = m''$, then $n'' = \frac{3m-a}{2} - m - 1 = \frac{m-a-2}{2}$, $n' = \frac{m+a-2}{2}$, $n = \frac{m+a-2}{2}$.

If $m$ is even, then $a$ may be put $= 0$, and the Value of $G$ will be found $= \frac{1}{4}m^2$; consequently, less than would result from the Combination of the Equations two by two, we may also put $a = 2$, except when $m = 2$, and then $G = \frac{3m^2+1}{4}$.

If $m$ is odd, put $a = 1$, and there will result $G = \frac{3m^2+1}{4}$ which Value is leſs than would reſult from the Combination of the Equations two by two; now as there is no Reaſon why the equal Values of $n$ and $n'$ ſhould be employed with Reſpect to two of three Equations, rather than with Reſpect of the two others, we are to make Uſe of them both Ways, and there will reſult two Equations in $y$ and $z$, each of a Degree expreſſed by $\frac{3m^2+1}{4}$.

Rules derived from the precedent principles for exterminating an unknown quantity of ſeveral dimenſions in three equations.

Hence, if there are three Equations given each of a Degree $m$ in which the higheſt Dimenſion of $x$ is expreſſed by $m$, if $m$ be an even Number we are to multiply each Equation by an undetermined Multinomial of a Degree expreſſed by $\frac{m-2}{2}$, and by Means of the Equations reſulting from the Suppoſition that the Powers of $x$ in the Sum of the three Products ſhould be deſtroyed, an Equation in $y$ and $z$ will be obtained of a Degree expreſſed by $\frac{3}{4}m^2$. To obtain another Equation, we are to multiply two of the three propoſed Equations by a Multinomial of a Degree expreſſed by $\frac{m}{2}$, and the third by a Multinomial of a Degree expreſſed by $\frac{m-4}{2}$, and by putting in the Sum of the three Products the Terms including the Powers of $x = 0$, there will reſult an Equation in $y$ and $z$ of a Degree expreſſed by $\frac{3}{4}m^2+1$.

But if $m$ be an odd Number, we are to multiply the firſt and ſecond Equation by a Multinomial of a Degree expreſſed by $\frac{m-1}{2}$, and the third by a Multinomial of a Degree expreſſed by $\frac{m-3}{2}$, we are to multiply alſo the firſt and third Equations, each by a Multinomial of a Degree expreſſed by $\frac{m-1}{2}$, and the ſecond by a Multinomial of a Degree expreſſed by $\frac{m-3}{2}$, or elſe the ſecond and third by a Multinomial of a Degree $\frac{m-1}{2}$, and the firſt by a Multinomial of a Degree $\frac{m-3}{2}$, and there will reſult two Equations in $y$ and $z$, each of a Degree expreſſed by $\frac{3m^2+1}{4}$.

LXVII.

From whence we may perceive, that by combining the Equations two by two, there will reſult an Equation in $y$ and $z$ of higher Dimenſions

Inconveniency to which the method of exterminating an unknown quantity by comparing the equations two by two is liable.

than when the three Equations are combined together; for by the first Method the resulting Equation will rise, in general to a Number of Dimensions expressed by $m^4$, and by the second to $\frac{1}{4}m^2(\frac{1}{4}m^2+1)^2$ when $m$ is an even Number, and to $\left(\frac{3m^2+1}{4}\right)^2$ Dimensions, when $m$ is odd. This Difference is rendered more sensible by the following Table.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| By 1st. Method, | $m=$ | 2 | 3 | 4 | 5 | 6 | 7 |
| | $G=$ | 16 | 81 | 256 | 625 | 1296 | 2401 |
| By 2d. Method, | $G=$ | 12 | 49 | 156 | 361 | 756 | 1369 |

LXVIII.

2°. Let $m=6$, $m'=5$, $m''=3$, $p=4$, $p'=5$, $p''=19$, there will result $n''=\frac{-4-a}{2}$, $n'=\frac{12+a}{2}$, $n=\frac{10+a}{2}$.

The least Value that can be given to $a$ is $a=-4$, and then $G=83$. But if we invert the Order of the Equations, and write down $m=6$, $m'=3$, $m''=5$, $p=4$, $p'=19$, $p''=5$, there will result $n''=\frac{20-a}{2}$, $n'=\frac{a-12}{2}$, $n=\frac{a-18}{2}$, the least Value that can be given to $a$ is $a=18$, and then $G=104$.

Inverting again the Order of the Equations, and writing down $m=5$, $m'=3$, $m''=6$, $p=5$, $p'=19$, $p''=4$, there will result $n''=\frac{20-a}{2}$, $n'=\frac{a-14}{2}$, $n=\frac{a-18}{2}$, the least Value that can be given to $a$ is still $a=18$, and then $G$ will be found $=85$; but this Combination cannot take Place because it gives $m+n<m''+n''$ &c. and if the Equations be combined two by two, there will be found $G=80$, $G=144$, and $G=125$. Wherefore, in this Case, one of the Equations in $x$ and $y$ is found by combining the Equations together, and the other by combining two of them together; we are to multiply the Equations indicated by $m=6$, $m'=5$, $m''=3$; I say, multiply those Equations by the Multinomials indicated by $n=3$, $n'=4$, $n''=0$, and the two Equations to be combined to obtain the second Equation in $y$ and $z$, are those indicated by $m=6$, $m'=5$.

LXIX.

3°. Let $m=7$, $m'=6$, $m''=5$, $p=3$, $p'=4$, $p''=2$, there will result $n''=\frac{n-a}{2}$, $n'=\frac{a-1}{2}$, $n=\frac{a-3}{2}$, the least Value therefore that can be given to $a$ is $a=3$; but it is useless, because it gives $m+n<m''-n''$; let then $a=5$, and $G$ will be found $=52$; changing the Order of the Equations, and putting $m=7$, $m'=5$, $m''=6$, $p=3$, $p'=2$, $p''=4$, there will result $n''=\frac{5-a}{2}$,

$n' = \frac{5+a}{2}$, $n = \frac{1+a}{2}$, the leaſt Value that can be given to $a$ is $a = -1$, but it is uſeleſs, becauſe it gives $m+n < m''+n''$. Let $a = 1$ there will reſult $G = 52$, as it ſhould be, ſince the Multiplicators of the Equations will be the ſame in this Caſe as in the foregoing. Let then $a = 3$ there will reſult $G = 54$, changing again the Order of the Equations, and writing down $m = 6$, $m' = 5$, $m'' = 7$, $p = 4$, $p' = 2$, $p'' = 3$; there will reſult $n'' = \frac{5-a}{2}$, $n' = \frac{3+a}{2}$, $n = \frac{1+a}{2}$; the Suppoſitions of $a = -1$, $a = 1$ cannot take Place. Let then $a = 3$, then $G = 52$, which give the ſame Multiplicators as were found above, and for the ſame Equations. If $a$ be put equal to 5, then $G = 56$, combining the Equations two by two, there reſults $G = 88$, $G = 64$, $G = 62$. Wherefore, the two Equations in $y$ and $z$ will be found by combining the three Equations together, the one being of 52 Dimenſions, and the other of 54.

LXX.

If four Equations are given involving four unknown Quantities, to exterminate one of them as $x$; let $m, m', m'', m'''$, denote the Exponents of $x$ in the four Equations, let the Dimenſions of the Coeficients of the ſeveral Powers of $x$ in the firſt Equation be expreſſed by $p, p+1, p+2, p+3$, &c. in the ſecond by $p', p'+1, p'+2, p'+3$, &c. in the third by $p'', p''+1, p''+2$, &c. and in the fourth by $p''', p'''+1, p'''+2, p'''+3$, &c.

How an unknown quantity of ſeveral dimenſions is exterminated out of four equations by the foregoing method.

Let $n, n', n'', n'''$, denote the Exponents of the undetermined Multinomials by which the Equations being multiplied, in the Sum of the four Products the Powers of $x$ will be deſtroyed.

By a ſimilar Reaſoning to that employed above, it will appear, that $m+n$ ſhould be put equal to $m'+n'$, and $m+n+1 = n+1+n'+1+n''+1+n'''+1$; from whence is deduced $n = m'-n''-n'''-3$ and $n' = m-n''-n'''-3$; alſo $m+n$ ſhould be $> m''+n''$ or at furtheſt equal, and $m+n > m'''+n'''$, or at furtheſt equal.

It will appear likewiſe that $S = (2p-n)\left(\frac{n+1}{2}\right) + (2p'-n')\left(\frac{n'+1}{2}\right) + (2p''-2m-2n+2m''+n'')\left(\frac{n''+1}{2}\right) + (2p'''-2m-2n+2m'''+n''')\left(\frac{n'''+1}{1}\right)$,

Conſequently, $G = (m+n+1)\left(\frac{m+n}{2}\right) + (2p-n)\left(\frac{n+1}{2}\right) + (2p'-n')\left(\frac{n'+1}{2}\right) + (2p''-2m-2n+2m''+n'')\left(\frac{n''+1}{2}\right)$

$+ (2p''' - 2m - 2n + 2m''' + n''') \left(\frac{n''' + 1}{2}\right)$,

or ſubſtituting for $n$ and $n'$ their Values,

$G = m\,m' + p\,m' + p'\,m - 2m - 2p - 2m' - 2p' + m'' + p'' + m''' + p''' + 3 - n''\,n''' - (m + m' - m'' + p + p' - p'' - n'' - n''' - 3)\,n'' - (m + m' - m''' + p + p' - p''' - n'' - n''' - 3)\,n'''$.

To find the Values of $n''$ and $n'''$ which ſubſtituted in the preceding Expreſſion of $G$ will render it the leaſt poſſible, we are to find, 1°. The limiting Ratio of the cotemporary Increments of $G$ and $n''$, conſidered as the only Quantities undetermined in the preceding Equation, and to put this Limit equal to Nothing; then to find the limiting Ratio of the cotemporary Increments of $G$ and $n'''$, ſuppoſed to be the only Quantities undetermined in the ſame Equation, and to put this Limit alſo equal to Nothing. Theſe Operations being performed you will find,

$n'' = \frac{m + m' + m'' + m''' + p + p' + p'' + p'''}{3} - m'' - p'' - 1$, and

$n''' = \frac{m + m' + m'' + m''' + p + p' + p'' + p'''}{3} - m''' - p''' - 1$,

but becauſe $n$, $n'$, $n''$, $n'''$, ſhould be poſitive integer Numbers, and anſwer the Conditions mentioned above.

We muſt put $n'' = \frac{m + m' + m'' + m''' + p + p' + p'' + p''' - a}{3} - m'' - p'' - 1$,

and $n''' = \frac{m + m' + m'' + m''' + p + p' + p'' + p''' - b}{3} - m''' - p''' - 1$,

$a$ and $b$ expreſſing the leaſt Numbers that can anſwer thoſe Conditions.

General expreſſion of the degree of the reſulting equations after an unknown quantity of ſeveral dimenſions is exterminated out of four equations.

Subſtituting in $G$ thoſe Values of $n''$ and $n'''$, there will reſult

$G = m\,m' + m\,m'' + m\,m''' + m\,p' + m\,p'' + m\,p''' + m'\,m'' + m'\,m''' + m'\,p + m'\,p'' + m'\,p''' + m''\,m''' + m''\,p + m''\,p' + m''\,p''' + m'''\,p + m'''\,p' + m'''\,p'' + p\,p'' + p\,p''' + p'\,p'' + p'\,p''' + p''\,p''' + \frac{a^2 + ab + b^2}{9} - 3\left(\frac{m + m' + m'' + m''' + p + p' + p'' + p'''}{3}\right)^2$.

LXXI.

Let $p = p' = p'' = p''' = 0$, $m = m' = m'' = m'''$, then

$n''' = \frac{m - b - 3}{3}$, $n'' = \frac{m - a - 3}{3}$, $n' = \frac{m + a + b - 3}{3}$,

$n = \frac{m + a + b - 3}{3}$.

It is eaſy to perceive firſt, that whatever Values poſitive or negative are given to $a$ and $b$, if $m$ be leſs than 3, it will not be poſſible to render $n$, $n'$, $n''$, $n'''$, at the ſame Time poſitive. In effect if $m = 2$, as in this Caſe, $x^2$ and $x$ are the only Quantities to be exterminated, the

Equations are to be combined three by three two different Ways, and the third Equation will be obtained by combining two of them together.

If $m = 3$, then by putting $a = 0$, and $b = 0$, positive Values will be obtained for $n$, $n'$, $n''$, $n'''$, consequently the four Equations may be combined together; the two other Equations without $x$ will be obtained by combining the Equations three by three, two different Ways.

But when $m$ is greater than 3, there are three Cases to be considered, either $m$ is exactly divisible by 3, or after the Division there remains 1, or there remains 2.

Rules derived from the preceding principles for exterminating an unknown quantity of several dimensions in four equations.

First Case. To obtain in this Case the first Equation without $x$, put $a = 0$, $b = 0$; to obtain the second, put $a = 3$, $b = 0$; and to obtain the third, put $a = 0$, and $b = 3$. The corresponding Values of $G$ will be $G = \frac{2}{3}m^2$, $G = \frac{2}{3}m^2 + 1$, $G = \frac{2}{3}m^2 + 1$. From whence it is easy to conclude that the Degree of the resulting Equation after all the unknown Quantities are exterminated, will be expressed by $(\frac{1}{3}m^4 + \frac{2}{3}m^2)(\frac{1}{3}m^4 + \frac{2}{3}m^2 + 1)$.

Second Case. To obtain the first Equation without $x$, put $a = 1$, $b = 1$, and because by changing the Order of the Equations, there will result a different Factor for the same Equation, the Supposition of $a = 1$ and $b = 1$ alone will furnish the three Factors to be applied to each Equation to obtain the three Equations without $x$. For Example, if $m = 4$ then $n''' = 0$, $n'' = 0$, $n' = 1$, $n = 1$, and $m$, $m'$, &c. being all equal, it is indifferent to which of the four Equations the Values of $n$ or of $n'$ are referred, consequently there may be formed three Combinations, each of which will furnish a different Equation. The Value of $G$ will be the same in each Case and $= \frac{2m^2 + 1}{3}$, from whence it is easy to conclude that the Degree of the resulting Equation after all the unknown Quantities are exterminated will be expressed by $\left(\frac{m^4 + m^2 + 1}{3}\right)^2$.

Third Case. The three Equations without $x$ will be obtained by putting $a = 2$, $b = 2$, and by applying the Factors three different Ways to the proposed Equations, each of which will rise to a Degree expressed by $\frac{2m^2 + 4}{3}$, from whence it is easy to conclude that the Degree of the resulting Equation will be $\left(\frac{m^4 + 4m^2 + 4}{3}\right)\left(\frac{m^4 + 4m^2 + 7}{3}\right)$.

Let now $p = 2$, $p' = 0$, $p'' = 1$, $p''' = 1$, $m = 7$, $m' = 6$, $m'' = 5$, $m''' = 4$; we will have $n''' = \frac{8 - b}{3}$, $n'' = \frac{5 - a}{3}$, $n' = \frac{a + b - 1}{3}$, $n = \frac{a + b - 4}{3}$. Let $b = -1$ and $a = 5$, we will have $n''' = 3$, $n'' = 0$, $n' = 1$, $n = 0$, and $G = 26$. If we put $a = -1$, $b = 5$,

there will result $n''' = 1$, $n'' = 2$, $n' = 1$, $n = 0$, and $G = 26$. If we put $b = 2$, $a = 2$, then $n''' = 2$, $n'' = 1$, $n' = 1$, $n = 0$, and $G = 25$. Now whatever Alteration is made in the Order of the Equations, there will result the ſame Values for $G$, if not greater; it remains therefore to examine whether more ſimple Values may not be obtained for $G$, by combining the Equations three by three, or two by two. The Calculation being performed, it will appear that the Equations indicated by $m' = 6$, $m'' = 5$, $m''' = 4$, $p' = 0$, $p'' = 1$, $p''' = 1$, will give $G = 24$, and thoſe indicated by $m = 7$, $m'' = 5$, $m''' = 4$, $p = 2$, $p'' = 1$, $p''' = 1$, will give $G = 25$; wherefore in order to obtain the three ſimpleſt Equations that can reſult after $x$ is exterminated, we are to compare the four propoſed Equations three by three, the two different Ways, juſt mentioned; and the four together after the Manner indicated by the Combination which gave $G = 25$. The Combinations of the Equations two by two, all give Values for $G$ greater than thoſe already found.

LXXII.

How an unknown quantity of ſeveral dimenſions is exterminated out of five equations by the foregoing method.

If five Equations are given involving five unknown Quantities, to exterminate one of them as $x$, let $m$, $m'$, $m''$, $m'''$, $m''''$, denote the Exponents of the Degree of $x$ in the propoſed Equations, and let $p, p', p'', p''', p''''$ expreſs the Dimenſions of the firſt Coeficient of thoſe Equations, the Dimenſions of the other Coeficients being ſuppoſed to obſerve the ſame Law as heretofore; by a ſimilar Reaſoning to that employed for 2, 3, 4 unknown Quantities, it will appear that $m + n = m' + n'$, $m + n + 1 = n + 1 + n' + 1 + n'' + 1 + n''' + 1 + n'''' + 1$, from whence is deduced $n = m' - n'' - n''' - n'''' - 4$, and $n' = m - n'' - n''' - n'''' - 4$.

It will be demonſtrated in like Manner that

$$G = (m + n + 1)\left(\frac{m+n}{2}\right) + (2p - n)\left(\frac{n+1}{2}\right) + (2p' - n')\left(\frac{n'+1}{2}\right)$$
$$+ (2p'' - 2m - 2n + 2m'' + n'')\left(\frac{n''+1}{2}\right) + (2p''' - 2m - 2n + 2m''' + n''')\left(\frac{n'''+1}{2}\right)$$
$$+ (2p'''' - 2m - 2n + 2m'''' + n'''')\left(\frac{n''''+1}{2}\right) = mm' + pm' + p'm - 3m - 3p - 3m'$$
$$- 3p' + m'' + p'' + m''' + p''' + m'''' + p'''' - n''n''' - n''n'''' - n'''n'''' + 6 - (p + p'$$
$$- p'' + m + m' - m'' - n'' - n''' - n'''' - 4)n'' - (p + p' - p''' + m + m' - m''' - n''$$
$$- n''' - n'''' - 4)n''' - (p + p' - p'''' + m + m' - m'''' - n'' - n''' - n'''' - 4)n''''.$$

The Condition that $G$ ſhould be the leaſt Number poſſible, requires that we ſhould put equal to nothing, 1°. The limiting Ratio of the cotemporary Increments of $G$ and $n''$, conſidered as the only Quantities undetermined in the preceding Expreſſion. 2°. The limiting Ratio of the cotemporary Increments of $G$ and $n'''$, conſidered as the only undetermined Quantities in the foregoing Expreſſion. 3°. The limiting

Ratio of the cotemporary Increments of $G$ and $n''''$, conſidered as the only undetermined Quantities in the aforeſaid Expreſſion. Theſe Operations being performed, we will have

$$n'' = \frac{m+m'+m''+m'''+m''''+p+p'+p''+p'''+p''''}{4} - m'' - p'' - 1,$$

$$n''' = \frac{m+m'+m''+m'''+m''''+p+p'+p''+p'''+p''''}{4} - m''' - p''' - 1,$$

$$n'''' = \frac{m+m'+m''+m'''+m''''+p+p'+p''+p'''+p''''}{4} - m'''' - p'''' - 1,$$

but becauſe $n''$, $n'''$, $n''''$ ſhould be integer Numbers and poſitive, and ſuch that $m+n > m''+n''$, $m+n > m'''+n''''$, $m+n > m''''+n''''$, or at the furtheſt equal.

I put $n'' = \frac{m+m'+m''+m'''+m''''+p+p'+p''+p'''+p''''-a}{4} - m'' - p'' - 1,$

$$n''' = \frac{m+m'+m''+m'''+m''''+p+p'+p''+p'''+p''''-b}{4} - m''' - p''' - 1,$$

$$n'''' = \frac{m+m'+m''+m'''+m''''+p+p'+p''+p'''+p''''-c}{4} - m'''' - p'''' - 1,$$

$a$, $b$, $c$ being the leaſt Numbers poſitive or negative that can anſwer thoſe Conditions.

If we ſubſtitute in $G$ thoſe Values of $n''$, $n'''$, $n''''$, we will have

$$G = m\,m' + m\,m'' + m\,m''' + m\,m'''' + m\,p' + m\,p'' + m\,p''' + m\,p'''' + m'\,m'' + m'\,m''' + m'\,m'''' + m'\,p + m'\,p'' + m'\,p''' + m'\,p'''' + m''\,m''' + m''\,m'''' + m''\,p + m''\,p' + m''\,p''' + m''\,p'''' + m'''\,m'''' + m'''\,p + m'''\,p' + m'''\,p'' + m'''\,p'''' + m''''\,p + m''''\,p' + m''''\,p'' + m''''\,p''' + p\,p' + p\,p'' + p\,p''' + p\,p'''' + p'\,p'' + p'\,p''' + p'\,p'''' + p''\,p''' + p''\,p'''' + p'''\,p'''' + \frac{a^2+b^2+c^2+ab+ac+bc}{16} - 6\left[\frac{m+m'+m''+m'''+m''''+p+p'+p''+p'''+p''''}{4}\right]^2.$$

General expreſſion of the degree of the reſulting equations after an unknown quantity of ſeveral dimenſions is exterminated out of five equations.

LXXIII.

Let $p = p' = p'' = p''' = p'''' = 0$, $m = m' = m'' = m''' = m''''$, then $n'''' = \frac{m-c-4}{4}$, $n''' = \frac{m-b-4}{4}$, $n'' = \frac{m-a-4}{4}$, $n' = \frac{m+a+b+c-4}{4}$, $n = \frac{m+a+b+c-4}{4}$.

Whence if $m$ is leſs than 4, there are no Values which ſubſtituted for $a$, $b$, $c$ will render $n$, $n'$, $n''$, $n'''$, $n''''$ poſitive, that is, the unknown Quantity is to be exterminated by the Rules laid down for a leſſer Number of unknown Quantities.

If $m = 4$, the unknown Quantity may be exterminated by combining the five Equations together but only one Way, by putting $a = b = c = 0$.

Rules derived from the

preceding principles for extefminating an unknown quantity of feveral dimenfions in five equations.

If $m$ be greater than 4, there are four Cafes to be conſidered, either $m$ is exactly diviſible by 4, or after the Diviſion there remains 1 or 2, or in fine 3.

First Case. The firſt Equation without $x$ will be obtained by putting $a = 0$, $b = 0$, $c = 0$, the ſecond by putting $a = 0$, $b = 0$, $c = 4$, the third by putting $a = 0$, $b = 4$, $c = 0$, in fine, the fourth by putting $a = 4$, $b = 0$, $c = 0$. The correſponding Values of $G$ will be $G = \frac{5}{8} m^2$, $G = \frac{5}{8} m^2 + 1$, $G = \frac{5}{8} m^2 + 1$, $G = \frac{5}{8} m^2 + 1$.

Second Case. Put $a = 1$, $b = 1$, $c = 1$, and the Values which reſult for $n$, $n'$, &c. being not all of the ſame Degree, they may be applied four different Ways to the five Equations, whereby the four Equations without $x$ will be obtained each of a Degree, $G = \frac{5m^2 + 3}{8}$.

Third Case. Put $a = 2$, $b = 2$, $c = 2$; proceeding as in the ſecond Caſe, the four Equations without $x$ will be obtained; which will riſe each to a Degree expreſſed by $\frac{5m^2}{8} + \frac{3}{2}$.

Fourth Case. Put $a = 3$, $b = 3$, $c = 3$, and proceeding as in the two foregoing Caſes, the four Equations without $x$ will be obtained, each of a Degree expreſſed by $\frac{5}{8} m^2 + \frac{27}{8}$.

LXXIV.

In general if there are $N$ unknown Quantities and $N$ Equations given; from all that precedes it is eaſy to conclude,

1°. That if $A$ denotes the Sum of the Quantities $m$, $m'$, $m''$, &c. and $p$, $p'$, $p''$, &c. Then

General expreſſion of the degree of the reſulting equations after an unknown quantity of ſeveral dimenſions is exterminated out of N equations.

$$n'' = \frac{A - a}{N - 1} - m'' - p'' - 1, \quad n''' = \frac{A - b}{N - 1} - m''' - p''' - 1,$$

$$n'''' = \frac{A - c}{N - 1} - m'''' - p'''' - 1, \quad n''''' = \frac{A - d}{N - 1} - m''''' - p''''' - 1,$$

and ſo on; likewiſe $n = \frac{A + a + b + c + d +, \&c.}{N - 1} - m - p - p' - 1$,

$$n' = \frac{A + a + b + c + d +, \&c.}{N - 1} - m' - p - p' - 1.$$

2°. That if $B$ denotes the Sum of the Squares together with the Sum of the Products of the Quantities $a$, $b$, $c$, &c. taken two by two, and $C$ the Sum of the Products of the Quantities $m, m', m'', \&c.$ $p, p', p'', p''', \&c.$ multiplied two by two, omitting the Products $mp$, $m'p'$, $m''p''$, &c. and the Product $pp'$ of the two Quantities, which belong to the Equations in which $m + n$ was ſuppoſed equal to $m' + n'$, there will reſult in general

$$G = C + \frac{B}{(N-1)^2} - (N-1)\frac{N-2}{2}\left(\frac{A}{N-1}\right)^2 = C + \frac{B}{(N-1)^2} - \frac{N-2}{2(N-1)} \times A^2.$$

## CHAP. III.

*Of the Resolution of Equations of all Degrees which consist of two Terms only, of those consisting of three Terms that can be reduced by the Method of Equations of the second Degree, to Equations consisting of two Terms: with different Operations relative to those Equations, such as the raising of Quantities to any Power, the Extraction of Roots, the Reduction of radical Quantities, &c. and of the Reduction of Equations by surd Divisors.*

HAVING fully treated of Problems of the second Degree, we shall now proceed to explain how to solve Problems of higher Degrees, beginning with those which produce Equations consisting of two Terms; of which Kind is the following, which in its full Extent produces Equations of every Degree consisting of two Terms.

I.

Problem which in its full extent produces equations of every degree consisting of two terms.

*A young Gentleman borrowed a Sum of Money a, for the Loan of which he was to allow Interest payable at stated Terms, but it not suiting his Convenience to pay it at the End of the first Term, the Lender agreed that it should be continually added to the Principle after it came due, so that at the End of each Term, the Sum or Amount should become a new Principle for the succeeding Term, until the Whole should be discharged at one Payment. Now after a certain Time t, the Debt amounted to a Sum r. The Rate of Interest allowed is required?*

Let the arbitrary Number (but commonly 100) that the Rate of Interest is computed from, be expressed by $d$, and the Interest of this Number for one Term by $i$. Now the Principal of the first Term being $a$, its Interest will be $\frac{ai}{d}$; to which $a$ or $\frac{ad}{d}$ being added, the Sum due at the End of the first Term will be $\frac{ad+ai}{d} = a \times \frac{d+i}{d}$; the Principal of the second Term being $\frac{ad+ai}{d}$, its Interest will be $\frac{aid+ai^2}{d^2}$, to which the Principal reduced to the same Denominator, $d^2$ being added, the Sum due at the End of the second Term will be $\frac{ad^2+2aid+ai^2}{d^2} = a \times \left(\frac{d+i}{d}\right)^2$. In like Manner it will appear that the Sum due at the End of the third Term will be $\frac{ad^3+3aid^2+3ai^2d+ai^3}{d^3} = a \times \left(\frac{d+i}{d}\right)^3$. Without proceeding any further it is easy to perceive that those different Results form a Series of Quantities which increase by one common Multiplicator, or in

the Language of the Analysts, a geometrical Progression; the first Term being $a$, and the common Multiplicator $\frac{d+i}{d}$, which for brevity Sake I shall denote by $x$. The Term of the Progression where $x$ is raised to the Power whose Exponent is 1, expresses the Sum due at the End of the first Term, that where $x$ is raised to the Power whose Exponent is 2, the Sum due at the End of the second Term, and in general the Term of the Progression where $x$ is raised to the Power whose Exponent is $t$, expresses the Sum $r$, due at the Expiration of the Time $t$; whence we have $r = a\,x^t$, which Equation being solved, the Value of $x$, and consequently the Rate of Interest $i = x\,d - d$ will be obtained.

II.

Equations of the third degree consisting of two terms.

Let us first suppose the Equation to be of the third Degree, that is, $a\,x^3 = r$, to solve this Equation, it is easy to perceive that it suffices to take away from $x^3$ its Coeficient, and extract the cube Root of the two Members. The Character employed by the Analysts to express the cube Root, is the same as for the square Root, with 3 put over it.

The figure 3 is put over the character $\surd$ to denote the cube root.

Thus to express the Value of $x$ deduced from the Equation $a\,x^3 = r$, or $x^3 = \frac{r}{a}$, they write down $x = \sqrt[3]{\frac{r}{a}}$. If, for Example, the Sum lent $a$ be £1000 and remained unpaid 3 Years amounted to £1331, then the Amount of £1 in one Year, or $x = \sqrt[3]{1,331}$.

III.

It is to be observed that we are not at Liberty to prefix the Sign + or − to the radical Sign of the cube Root, as to that of the square Root, but that the Sign of the cube Root is the same as the Sign of the Quantity whose Root is required, because the Cube of a positive Quantity is positive, and that of a negative Quantity negative. However, though a Cube has only one Root, and this Root has only one Sign, we are not to conclude that an Equation as $a\,x^3 = r$, will give only one Value for $x$. That this may appear, let the Equation $a\,x^3 = r$ be reduced to this Form $x^3 - \frac{r}{a} = 0$, and its Root $x = \sqrt[3]{\frac{r}{a}}$, also to this Form $x - \sqrt[3]{\frac{r}{a}} = 0$, then dividing $x^3 - \frac{r}{a}$ by $x - \sqrt[3]{\frac{r}{a}}$, the Quotient will be an Equation of the second Degree, which will contain the other two Roots.

A radical cube can have but one sign prefixed to it.

To perform this Operation with less Trouble, I put $\frac{r}{a} = c^3$, whereby the preceding Equations are transformed into $x^3 - c^3 = 0$, and $x - c = 0$, dividing the first by the second, the Quotient will be found to be $x^2 + c\,x + c^2 = 0$, whose two Roots are expressed by $x = -\frac{1}{2}c \pm \sqrt{-\frac{3}{4}c^2}$, and will become the second and third Values

of $x$ in the Equation $x^3 = \frac{b}{a}$ when for $c$ is ſubſtituted its Value $\sqrt[3]{\frac{b}{a}}$.

IV.

The Subſtitution being made, the two Values of $x$ will be found to be $-\frac{1}{2}\sqrt[3]{\frac{b}{a}} \pm \sqrt{\left(-\frac{3}{4}\sqrt[3]{\frac{bb}{aa}}\right)}$ for it is eaſy to perceive that the Square of $\sqrt[3]{\frac{b}{a}}$ that is, the product of $\sqrt[3]{\frac{b}{a}}$ into $\sqrt[3]{\frac{b}{a}}$ ſhould be $\sqrt[3]{\frac{bb}{aa}}$, and that in general the Multiplication of cube Roots, as that of ſquare Roots, is performed by multiplying firſt the Quantities which are under the radical Sign, and afterwards prefixing this Sign to their Product.

How cubical radicals are multiplied into one another

V.

The three Roots of the propoſed Equation $a x^3 = b$, are therefore $x = \sqrt[3]{\frac{b}{a}}$, $x = -\frac{1}{2}\sqrt[3]{\frac{b}{a}} + \sqrt{\left(-\frac{3}{4}\sqrt[3]{\frac{bb}{aa}}\right)}$, $x = -\frac{1}{2}\sqrt[3]{\frac{b}{a}} - \sqrt{\left(-\frac{3}{4}\sqrt[3]{\frac{bb}{aa}}\right)}$ the firſt real, and the two others imaginary, which however may be ſaid to reſolve the propoſed Equation.

Roots of an equation of the third degree conſiſting of two terms.

VI.

In like Manner an Equation conſiſting of two Terms of any Degree may be ſolved, by employing a radical having for Exponent the Exponent of the unknown Quantity in the propoſed Equation. For Example, the Equation $a x^t = r$ or $x^t = \frac{r}{a}$ being propoſed, there would reſult $x = \sqrt[t]{\frac{r}{a}}$.

Reſolution of equations conſiſting of two terms of every degree.

If $t$ is an odd Number, this Quantity is neceſſarily negative or poſitive, according as $\frac{r}{a}$ is negative or poſitive: if $t$ is an even Number the Root will have as the ſquare Root the Sign $\pm$, and will be reel when $\frac{r}{a}$ is poſitive, but when $\frac{r}{a}$ is negative ($t$ being an even Number) the two Roots expreſſed by $\pm\sqrt[t]{\frac{r}{a}}$ will be both imaginary, whence all the Equations expreſſed in general by $x^t = \frac{r}{a}$ can have no more than two reel Roots, the other Roots being neceſſarily imaginary, conſequently the Problems which produce thoſe Equations can have but two Solutions.

Thoſe equations can only have two real roots,

For Example, if the Debt was £10000. and remained unpaid for four Years amounted to £20736. the Equation to be ſolved would be $x^4 = 2{,}0736$, the two real Roots of which are 1,2 and — 1,2, the two imaginary ones $+\sqrt{-144}$ and $-\sqrt{-144}$; however only one ſolves the Queſtion as it is propoſed, and gives 20 *per Cent.* for the Rate of Intereſt required. Example.

If the Debt was £100000. and remained unpaid for five Years amounted to £161051. the Equation to be ſolved in this Caſe would be $x^5 = 1{,}61051$, which has only one real Root + 1,1, the other Roots are found by reſolving the Equation $x^4 + 1{,}1x^3 + 1{,}21x^2 + 1{,}331x + 1{,}4641 = 0$ which ariſes by dividing the Equation $x^5 - 1{,}61051 = 0$ by $x - 1{,}1 = 0$; it is eaſy to perceive without reſolving this Equation that its Roots are all imaginary, for if any of them was real they would reſolve alſo the Equation $x^5 = 1{,}61051$, conſequently ſome other Number beſides 1,1 raiſed to the fifth Power would give 1, 61051. Another example.

VII.

To compleat the Reſolution of Equations conſiſting of two Terms, it remains to ſhew how the radical Quantities they produce are reduced to their loweſt Terms, or how their Roots are extracted when they are perfect Powers. Obſervations on the involution of quantities.

To enable the young Analyſts to diſtinguiſh thoſe Caſes it will be neceſſary to make ſome Obſervations on the Converſe of this Operation, that is, on the raiſing Quantities to any Power.

If a Quantity as $a\,b\,c\,d$ was propoſed to be raiſed to a Power $m$, it is eaſy to perceive that this Power would be $a^m\,b^m\,c^m\,d^m$.

If it was propoſed to raiſe to any Power a Quantity as $\frac{a\,b\,c}{d\,e}$ having a Diviſor, it is manifeſt that the Diviſor and Dividend muſt be raiſed to the propoſed Power, and if the Quantities had Coeficients they muſt likewiſe be raiſed to the ſame Power alſo, if the Factors of the given Quantity are raiſed to any Powers, they will then be raiſed to a new Power, whoſe Exponent will be the Product of the Exponent they had already into that of the Power to which they are propoſed to be raiſed, thus $\frac{2\,a^2\,b3}{3\,c^4}$ raiſed to the Power 3 will give $\frac{8\,a^6\,b9}{27\,c^{12}}$; $\frac{a^m\,b^r}{c^n}$ raiſed to the Power $q$ will give $\frac{a^{mq}\,b^{rq}}{c^{nq}}$; all which is very eaſy to conceive, and is a neceſſary Conſequence of this Principle, that a Quantity raiſed to any Power is the Reſult of the Multiplication of this Quantity into itſelf as many Times leſs one, as there are Units in the Exponent of the Power. Rule for involving ſingle quantities.

Example. If the Sum of £1000. was lent at 6 *per Cent. per Annum*, and remained unpaid 3 Years, and it were required to find the Sum due at the Ex-

piration of that Time, in this Case $a = 1000$, $d = 100$, $i = 6$, $t = 3$, $\frac{d+i}{d} = x = \frac{106}{100} = \frac{53}{50}$; consequently the Sum due $r = ax^t$ $= 1000 \times \left(\frac{53}{50}\right)^3 = 1000 \times \frac{148877}{125000} = \frac{148877}{125} = £1191. \frac{2}{125}$.

VIII.

It is now easy to perceive that the Converse of this Operation, that is the Extraction of Roots is performed by dividing the Exponents of the Parts or Factors of the Quantity by the Number that denominates the Root required, whether those Factors are in the Numerator or in the Denominator. Let it be proposed for Example to extract the cube Root of $\frac{a^3 b^6}{c^9}$, I divide the Exponents 3, 6, 9 by 3, and assume the Quotients 1, 2, 3 for the new Exponents of the same Letters, whence $\frac{a b^2}{c^3}$ will be the cube Root sought. Evolution of single quantities.

If the Quantity had a Coefficient its cube Root is also to be extracted; thus the cube Root of $\frac{8 a^3 b^6}{c^9}$ will be $\frac{2 a b^2}{c^3}$.

In like Manner the Root of the fourth Power extracted out of $\frac{16 a^4 b^8}{d^4 c^{12}}$ will be $\frac{2 a b^2}{d c^3}$.

In general, the Root $n$ of the Quantity $\frac{a^m b^r}{c^q}$ will be expressed by $\frac{a^{\frac{m}{n}} b^{\frac{r}{n}}}{c^{\frac{q}{n}}}$, let the particular Values of $m, r, q, n$ be what they will, whence in the Place of any Quantity that has a fractional Exponent we may substitute the Root of a Quantity whose Exponent is an Integer, by placing above the radical Sign the Denominator of this Fraction, and placing within the radical Sign the Quantity having for Exponent the Numerator of this Fraction, that is $\sqrt[n]{\frac{a^m b^r}{c^q}} = \frac{a^{\frac{m}{n}} b^{\frac{r}{n}}}{c^{\frac{q}{n}}}$. What fractional powers denote.

If for Example the Sum of £1000. was lent at $7\frac{1}{2}$ *per Cent. per Annum* and remained unpaid one Year, two Months and thirteen Days, and the Sum due at the Expiration of that Time be required. Example.

In this Case $a = 1000$, $d = 100$, $i = 7\frac{1}{2}$, $\frac{d+i}{d} = x = \frac{107\frac{1}{2}}{100}$ $= \frac{215}{200} = \frac{43}{40}$, $t = \frac{438}{365} = \frac{6}{5}$ Years, consequently the Sum

due $r = a\,x^t = 1000 \times \frac{43}{40}\Big]^{\frac{6}{5}} = 1000\ \sqrt[5]{\frac{43}{40}}\Big]^6$.

IX.

What negative powers denote.

If it was required to find what principal Sum $a$, let out at a certain Rate of Intereſt $i$, would amount to a Sum $r$ at the Expiration of a certain Time $t$, then $a = \frac{r}{x^t} = r\,x^{-t}$, thoſe two Expreſſions being equal, for if they be multiplied by the ſame Power of $x$ there will reſult the ſame Product, for Example, $x^{2t} \times r\,x^{-t} = r\,x^t = \frac{x^{2t} \times r}{x^t}$, and in general any Quantity placed in the Denominator of a Fraction may be tranſpoſed to the Numerator if the Sign of its Exponent be changed.

Example

To apply this general Solution to an Example, let it be required to find what principal Sum of Money lent out at 6 *per Cent per Annum.* would amount to £385 13*s.* 7½*d.* at the Expiration of 7 Years, here $r = 385{,}681.1$ $x = 1{,}06$ and $t = 7$, conſequently $a = \frac{r}{x^t} =$ £256 10*s.* When $t = 0$

what the power o denotes.

then $\frac{d+i}{d}$ or $x =$ is equal Unity, conſequently $x^0 = 1$.

X.

As the foregoing Reductions and Transformations are neceſſary in an infinite Number of Operations, to render the Application of them eaſy to Beginners, here follow ſome Examples.

Examples of the foregoing Reductions and Transformations.

$$\frac{a^2\,b^{-3}}{c\,a^{-1}} = a^3\,b^{-3}\,c^{-1} = \frac{a^3}{b^3\,c}, \quad \sqrt{\frac{a^{-2}\,b^2}{a\,b^{-1}}} = \sqrt{a^{-3}\,b^3} = \sqrt{\frac{b^3}{a^3}} = \frac{b^{\frac{3}{2}}}{a^{\frac{3}{2}}}, \quad \frac{a^{\frac{1}{3}}\sqrt{b^{-3}}}{\sqrt[3]{a^2}} = a^{\frac{1}{3}}\,b^{-\frac{3}{2}}\,a^{-\frac{2}{3}}$$

$$= a^{-\frac{1}{3}}\,b^{-\frac{3}{2}} = \frac{1}{a^{\frac{1}{3}}\,b^{\frac{3}{2}}} = \frac{1}{b\sqrt{b}\sqrt[3]{a}}, \quad \frac{\sqrt[3]{\frac{a^3\,c^5}{b}} \times \sqrt[4]{a\,c^3}}{\sqrt{\frac{a\,b^2}{c}}}$$

$$= a^{\frac{3}{2}}\,b^{-\frac{1}{3}}\,c^{\frac{5}{3}}\,a^{\frac{1}{4}}\,c^{\frac{3}{4}}\,a^{-\frac{1}{2}}\,b^{-1}\,c^{\frac{1}{2}}$$

$$= a^{\frac{2}{3}+\frac{1}{4}-\frac{1}{2}}\,b^{-\frac{1}{3}-1}\,c^{\frac{5}{3}+\frac{3}{4}+\frac{1}{2}} = a^{\frac{5}{12}}\,b^{-\frac{4}{3}}\,b^{\frac{35}{12}}$$

$$= \frac{a^{\frac{5}{12}}\,c^{\frac{35}{12}}}{b^{\frac{4}{3}}} = \frac{\sqrt[12]{a^5\,c^{35}}}{\sqrt[3]{b^4}} = \frac{c^2\sqrt[12]{a^5\,c^{11}}}{b\sqrt[3]{b}}.$$

XI.

When there is only a Part or Factor of the propoſed Quantity whoſe Root can be extracted, the Root of this Part is placed before the Sign and the other Factors under the Sign, let it be propoſed, for Example, to extract the Root of the fifth Power out of $\frac{32\,a^{10}\,b^8}{486\,c^7}$, which is compoſed of the Product of $\frac{32\,a^{10}\,b^5}{243\,c^5}$ into $\frac{b^3}{2\,c^2}$, the firſt Factor being the fifth Power of $\frac{2\,a^2\,b}{3\,c}$ and the other having no Root of the fifth Power I ſet down $\frac{2\,a^2\,b}{3\,c}\sqrt[5]{\frac{b^3}{2\,c^2}}$ for the Root required. Extraction of the roots of imperfect powers.

In like Manner the cube Root of $\frac{8\,a^3\,b+16\,a^3\,c}{54\,d}$ will be $\frac{2}{3}\,a\sqrt[3]{\frac{b+2\,c}{2\,d}}$, becauſe the Quantity $\frac{8\,a^3\,b+16\,a^3\,c}{54\,d}$ is the Product of $\frac{8\,a^3}{27}$ into $\frac{b+2\,c}{2\,d}$, the firſt of which Quantities is a perfect Cube, that of $\frac{2}{3}\,a$, and the ſecond has no cube Root. In like Manner $\sqrt[5]{\frac{32\,a^9+128\,a^6\,b^3-160\,a^5\,b^4}{3\,b^6}}=\frac{2\,a}{b}\sqrt[5]{\frac{a^4+4\,a\,b^3-5\,b^4}{3\,b}}$.

XII.

When the Quantity whoſe Root is propoſed to be extracted is compoſed as the foregoing ones of ſeveral Terms, and after having ſeparated from thoſe Terms ſuch Quantities as they all include that are perfect Powers of the ſame Name with the Root, it is ſuſpected that the Remainder is a perfect Power of ſome commenſurable Quantity compoſed of ſeveral Terms; the Operation to be performed to diſcover when this happens is ſomewhat more difficult, to find the Method to be purſued in this Operation, we ſhall make ſome Obſervations on the Converſe, that is, on the raiſing of compound Quantities to any Power, from whence we will derive Rules for extracting the Roots of thoſe kind of Quantities beginning by the cube Root.

XIII.

Let the ſimpleſt compound Quantity $u+z$ be aſſumed, and let it be raiſed firſt to the Cube, or third Power. Firſt by multiplying $u+z$ into itſelf we ſhall have $u\,u+2\,u\,z+z\,z$, which is the Square or ſecond Power of $u+z$, and this again multiplied by $u+z$ gives $u^3+3\,u\,u\,z+3\,u\,z\,z+z^3$ or the Cube of $u+z$, whence it appears, that a Quantity conſiſting of two Parts when raiſed to the Cube gives, 1°. The Cube of the firſt Part, 2°. The Triple of the Square of this firſt Part multiplied into the ſecond Part, 3°. The Triple of the firſt In what the cube of a binomial conſiſts.

Part multiplied into the Square of the ſecond, 4°. The Cube of the ſecond.

XIV.

Method of extracting the cube root of compound quantities.

The cube Root therefore of any compound Quantity will be diſcovered after this Manner, firſt look out for a Term which is a Cube and this Cube correſponding to $u^3$ place its Root in the Quotient, which will correſpond to $u$, then ſubtract its Cube from the propoſed Quantity, and divide the Remainder by the Triple of the Square of this Root, the Quotient will be the ſecond Member of the Root, and will correſpond to $z$; then multiply this Term by the Quantity correſponding to $3uu + 3uz + zz$, that is, by the Triple of the Square of the firſt Member, together with the Triple of the Product of the firſt Quantity into the ſecond, and the Square of the ſecond, ſubtract the Product from the aforeſaid Remainder, and if nothing remains then the cube Root of the Binomial correſponding to $u + z$ is obtained: If there ſtill remain ſeveral Terms, to diſcover whether the propoſed Quantity is not the Cube of a Trinomial, thoſe two Terms muſt be employed in the ſame Manner as the firſt was, to find the ſecond.

XV.

First example.

Some Examples will illuſtrate this Method. Let the cube Root of the Quantity $8y^6 + 60b^2y^4 + 150b^4y^2 + 125b^6$ be required.

$$\begin{array}{l|l} 8y^6 + 60y^4b^2 + 150b^4y^2 + 125b^6 & \left\{\begin{array}{l} 12y^4 + 30b^2y^2 + 25b^4 \\ 2y^2 + 5b^2 \end{array}\right. \\ -8y^6 & \\ \hline 60b^2y^4 + 150b^4y^2 + 125b^6 & \\ -60b^2y^4 - 150b^4y^2 - 125b^6 & \\ \hline 0 \quad 0 \quad 0 & \end{array}$$

I extract the cube Root of the firſt Term $8y^6$, and ſet down this Root $2y^2$ beſide the propoſed Quantity, and then write down the Cube of $2y^2$ under this firſt Term, obſerving to change its Sign, that is, prefixing to it the Sign —, the Subtraction or Reduction being performed, I ſet down the Remainder $60b^2y^4 + 150b^4y^2 + 125b^6$, and place above $2y^2$ the Triple of its Square, that is, $12y^4$, I then divide the firſt Term $60b^2y^4$ by $12y^4$, and place the Quotient $5b^2$ beſide $2y^2$, I then add to $12y^4$ the Product $30b^2y^2$ of the Triple of $2y^2$ into $5b^2$, and I add to thoſe two firſt Terms $25b^4$ the Square of $5b^2$, I then multiply $5b^2$ by thoſe three Terms, and I ſet down their Products with their Signs changed under the Quantity $60b^2y^4 + 150b^4y^2 + 125b^6$, and after the Reduction there being no Remainder left, I conclude that the propoſed Quantity is a perfect Cube, and that its Root is $2y^2 + 5b^2$.

XVI.

$$\begin{array}{l|l} x^6+6bx^5+21b^2x^4+44b^3x^3+63b^4x^2+54b^5x+27b^6 & 3x^4+12bx^3+21b^2x^2+18b^3x+9b^4 \\ -x^6 & 3x^4+6bx^3+4b^2x^2 \\ & x^2+2bx+3b^2 \end{array}$$

$$\begin{array}{l} 6bx^5+21b^2x^4+44b^3x^3 \\ -6bx^5-12b^2x^4-8b^3x^3 \end{array}$$

$$\begin{array}{l} 9b^2x^4+36b^3x^3+63b^4x^2+54b^5x+27b^6 \\ -9b^2x^4-36b^3x^3-63b^4x^2-54b^5x-27b^6 \end{array}$$

Second example.

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If the cube Root of $x^6+6bx^5+21b^2x^4+44b^3x^3+63b^4x^2+54b^5x+27b^6$ was required, which it is eaſy to perceive ſhould conſiſt of more than two Terms, by an Operation ſimilar to that employed in the foregoing Example, the two firſt Terms of the Root will be found to be $x^2+2bx$; but as there is a Remainder $9b^2x^4+36b^3x^3+63b^4x^2+54b^5x+27b^6$, I divide the firſt Term $9b^2x^4$ of this Remainder by $3x^4$ triple of the Square of $x^2$, becauſe this Term is the firſt of thoſe which ariſes by tripling the Square of the Quantity $xx+2bx$ which actually correſponds to the firſt Part (called *u* Art. XIII.) of the cube Root ſought, having divided $9b^2x^4$ by $3x^4$, I ſet down the Quotient $3b^2$ beſide $xx+2bx$, I then add together the Triple of the Square of $xx+2bx$, the Triple of the Product of $xx+2bx$ into $3b^2$ and the Square of $3b^2$ and there reſults $3x^4+12bx^3+21b^2x^2+18b^3x^1+9b^4$, which I multiply by $3b^2$, and I ſet down the Terms with their Signs changed under the Quantity $9b^2x^4+36b^3x^3+63b^4x^2+54b^5x+27b^6$, and as after the Reduction there is no Remainder left I conclude that $x^2+2bx+b^2$ is the cube Root of the propoſed Quantity.

XVII.

The cube Root of any Number may be found after the ſame Manner, if it be a Number under 1000 its neareſt cube Root is found by the following Table. Method of extracting the cube root of numbers.

| Cubes | 1 | 8 | 27 | 64 | 125 | 216 | 343 | 512 | 729 | 1000 |
|---|---|---|---|---|---|---|---|---|---|---|
| Roots | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

where it is eaſy to obſerve that the Cube of a ſimple Number cannot conſiſt of more than three Places of Figures, ſince the Cube of 10 the leaſt Number conſiſting of two Places is 1000, the leaſt Number conſiſting of four Places of Figures, that the Cube of a Number conſiſting of two Places of Figures cannot conſiſt of more than ſix Places, ſince the Cube of 100 the leaſt Number conſiſting of three Places of Figures is 1000000, the leaſt Number conſiſting of ſeven Places: Proceeding in this Manner it will appear that the Cube of any Number cannot conſiſt

of more than the Triple of its Places, and in general, *that the Power* p *of a Number consisting of* n *Places cannot consist of more than* p n *Places*; from whence it follows, that a Number consisting of less than four Places of Figures can have but one Figure in its Root, that a Number consisting of more than four but less than seven Places can have but two Figures in its Root, that a Number consisting of more than seven but less than ten Places of Figures can have but three Places of Figures in its Root and so on, taking for Limits the Numbers 1, 4, 7, 10, 13, &c. whose common Difference is 3, consequently the Number of Places which the cube Root of any Number consists of is discoverable by Inspection: How the Numbers corresponding to those Places are found we shall now proceed to explain.

XVIII.

The foregoing method explained by an example.

Let it be proposed to extract the cube Root of 5,305,472 as it consists of seven Places of Figures, I conclude that the Root will consist of three, to determine the Numbers corresponding to those Places, I express them by $x$, $y$ & $z$, consequently $x^3+3x^2y+3xy^2+y^3+3[x^2+2xy+y^2]z+3[a+y]z^2+z^3=5,305,472$: Hence the Difficulty is reduced to find the Terms of this Quantity in the given Number. It has been proved that the Product of two Numbers multiplied into one another has as many Places of Figures to the right Hand of it as there are to the right Hand both of the Multiplcand and Multiplicator, consequently the Product of three Numbers multiplied into one another will always have as many Places of Figures to the right Hand of it, as there are to the right Hand of the three Numbers multiplied together. Now the first Number $x$ being hundreds its Cube $x^3$ will have six Places of Figures to the right Hand of it, whence separating by a Line the six Figures 305472 to the right Hand of the proposed Number, The Cube of the unknown Number $x$ of Hundreds of the cube Root will be found in the Part 5 to the left Hand of the separating Line, and will be the greatest Cube contained in this Part 5, but 1 is the greatest Cube it contains, consequently 1 the cube Root of 1 is the Number of Hundreds of the Root which I therefore set down in the Quotient and deducting 1 from 5 the proposed Number is reduced to 4305472.

```
5,305,472(174,41,&c.
1
4305
 300
2100
1470
 343
3913
392472
 86700
346800
  8160
    64
355024
37448000
 9082800
36331200
   83520
      64
36414784
1033216000
 912460800
 912460800
     52320
         1
 912513121
 120702879, &c.
```

The Square of a Number of Hundreds which has four Places of Figures to the right Hand of it, being multiplied by a Number of Tens which has one Place of Figures to the right Hand of it, the Product will have five Places of Figures to the right Hand of it: wherefore the Term $3x^2y$, will be contained in the Part 43 which has five Figures to the right Hand of it. But the Product ariſing from the Multiplication of three times the Square of the Number of Hundreds multiplied into a Number of Tens, being divided by three Times the Square of the Number of Hundreds, will manifeſtly give the Number of Tens, whence the Number of Tens of the Root will be found by dividing 43 by 3, triple the Square of the Number 1 Hundred of the Root; to determine the Part of this Quotient to be employed, I obſerve that $3x^2y + 3xy^2 + y^3$, or three Times the Square of the Number of Hundreds of the Root multiplied into the Number of Tens, together with three Times the Square of the Number of Tens into the Number of Hundreds, and the Cube of the Number of Tens ſhould not exceed 4305, ſince the Cube of the Number of Tens ſhould have three Places of Figures to the right Hand of it: dividing therefore 43 by 3 or 4305 by 300, inſtead of the whole Quotient 14 reſulting from this Diviſion I find only 7 to anſwer the above Condition; I therefore ſet down 7 in the Root to the right Hand of the 1 Hundred, and deduct $300 \times 7 + 7 \times 7 \times 10 \times 3 + 7 \times 7 \times 7$ or $2100 + 1470 + 343$ or 3913 from 4305, and to the Remainder 392 I annex the remaining Figures of the propoſed Number, and there reſults 392472, which ſhould contain the Terms $3(x^2 + 2xy + y^2)z + 3(x + y)z^2 + z^3$, or three Times the Square of the 17 Tens multiplied by the unknown Number of Units, together with three Times the Square of the Number of Units, multiplied into the Number of Tens, and the Cube of the Number of Units: Now becauſe three Times the Square of the 17 Tens multiplied into the Number of Units, ſhould have two Figures to the right Hand, it will be contained in the Part 3924, which has two Figures to the right Hand of it, whence dividing 3924 by 867 three Times the Square of 17, or dividing 392472 by 86700 the Quotient 4 will be the Number of Units of the Root, which I ſet down to the right Hand of the 17 Tens already found, and deduct

$3 \times 170 \times 170 \times 4 + 3 \times 170 \times 16 + 64$, or $346800 + 8160 + 64$, that is, 355024 from 392472 and there remains 37448, whence 174 is the cube Root required of the greateſt Cube contained in the propoſed Number 5305472, which exceeds this Cube by 37448.

XIX.

Method of approximating to the cube root of numbers.

When a Number is not an exact Cube, as the foregoing one, we may however approximate to its cube Root to any Degree of Exactneſs; for Example, to extract the cube Root of 5305472, inſtead of 5305472

it is easy to perceive that we may take $\frac{5305472000000}{1000000}$, which is equal to it; observing that the Denominator should be a cube Number, then extracting the cube Root of the Numerator which may be found true to an Unit, and dividing this Root by 100 the cube Root of the Denominator; the cube Root of $\frac{5305472000000}{3000000}$ or of 5305472 will be obtained true to $\frac{1}{100}$: If we had wrote $\frac{5305472000000000}{1000000000}$, it easy to perceive that the Root would be found true to $\frac{1}{1000}$, and so on, whence in extracting the cube Root after the proposed Number is gone through, if there is a Remainder, the Operation being continued by adding Periods of Cyphers to that Remainder, the true Root will be be obtained in Decimals to any Degree of Exactness, as in the Example.

XX.

Process of the extraction of the cube root of numbers.

It is easy to perceive, that the cube Root of any Number may be discovered by the same Method of reasoning employed in the foregoing Example, the Process of which may be expressed thus; point every third Figure beginning with the Units Place, and by these Points the Number will be distinguished into as many Periods as there are Figures in the Root; then find by the Table of Cubes the greatest Cube in the first pointed Period of the proposed Number, the cube Root of which place in the Quotient to the right Hand, and subtract the Cube thereof from the first pointed Period, bringing down to the Remainder the second Period; then divide this Resolvend by thrice the Square of the first Quotient Figure, considered as expressing a Number of Tens, whereby the second Quotient Figure of the required Root will be found; continue this Method till all the Periods are brought down, and then, if the proposed Number be a perfect Cube, there will be no Remainder: but if something should remain, annex Cyphers three at a Time, and carry on the Extraction decimally to any proposed Degree of Exactness.

XXI.

As by raising the Binomial $u + z$ to the third Power Rules are obtained for extracting the cube Root of Quantities; in like Manner, by raising it to the fourth, fifth, &c. Powers, Rules may be derived for extracting the Roots of the fourth, fifth, &c. Powers of Quantities: But to comprehend under one general Rule, the Extraction of Roots, the Analysts have sought a general Expression for the Power $m$ of $u + z$.

Investigation of Sir Isaac Newton's theorem for raising a binomial to any power.

To find this general Value of $(u + z)^m$, or of $u + z$ multiplied into itself as often as there are Units less one in $m$; let us first examine the Product arising from the Multiplication of several binomial Factors $x + a$, $x + b$, $x + c$, $x + d$, &c. and endeavour to discover the Law observed in the Formation of all the Terms of this Product.

Multiplying all thoſe Factors into one another there reſults, $x^5+(a+b+c+d+e)x^4+(ab+ac+ad+ae+bc+bd+be+cd+ce+de)\,x^3+(abc+abd+abe+acd+ace+ade+bcd+bce+bde+cde)x^2+(abcd+abce+abde+acde+bcde)\,x+abcde$.

From the Inſpection of this Product it is plain, 1°. That the firſt Term is $x$ raiſed to a Power equal to the Number of Factors.

2°. That the ſecond Term contains $x$ raiſed to a Power leſs by an Unit, having for Coeficient the Sum of all the Letters $(a, b, c, d, e,)$.

3°. That the third Term is compoſed of $x$ raiſed to a Power leſs by two Units, with a Coeficient equal to the Sum of all the Products that can be made by multiplying any two of the Letters $(a, b, c, d,$ &c.) by one another.

4°. That the 4th Term includes $x$ raiſed to a Power leſs by three Units, with a Coeficient equal to the Sum of all the Products that can be made by multiplying into one another any three of the Letters $(a, b, c, d, e,$ &c.): and after the ſame Manner all the other Terms are formed.

Applying therefore thoſe Obſervations to the preſent Caſe, in which all the Factors are equal, and their Number is expreſſed in general by $m$, it will appear that the firſt Term will be $x^m$; that the ſecond Term will be $x^{m-1}$ multiplied by $ma$, ſince $b$, $c$, &c. are equal to $a$, and their Number is $m$; that the third Term will be $x^{m-2}$, with a Coeficient equal to $a^2$, repeated as often as there are Rectangles $ab$, $ac$, $bc$, &c. in the Coeficient of the third Term of the Product of the Factors $x+a$, $x+b$, $x+c$, &c. the Number of which is ſuppoſed to be $m$: ſince all the Products $ab$, $ac$, $bc$, &c. ſhould be all equal each to $a^2$, when $b$, $c$, &c. are equal to $a$: that the fourth Term will be $x^{m-3}$, with a Coeficient equal to $a^3$, repeated as often as there are Products $abc$, $abd$, $acd$, $bcd$, &c. in the Coeficient of the fourth Term of the Product of the Factors $x+a$, $x+b$, $x+c$, $x+d$, &c. whoſe Number is $m$; and after the ſame Manner all the other Terms are formed.

The Queſtion is therefore reduced to know how many different Products a Number $m$ of Letters will admit of, when taken two by two, three by three, &c. for ſuppoſing that thoſe Numbers were found, and were expreſſed by $A, B, C, D,$ &c. there would reſult $x^m+ma\,x^{m-1}+Aa^2x^{m-2}+Ba^3x^{m-3}+Ca^4x^{m-4}+Da^5x^{m-5}$, &c. for the Value ſought of $(x+a)^m$.

To find firſt how many different Products a Number $m$ of Letters $a$, $b$, $c$, $d$, &c. taken two by two will admit of, I obſerve that when all thoſe Products are formed, there will be twice as many Letters ſet down as there are Terms.

I obſerve next, that each of the Letters $a$, $b$, $c$, &c. ſhould be repeated the ſame Number of Times, and that each Letter, being multiplied by all the reſt and not by itſelf, can not be repeated more than $m-1$ Times;

wherefore the Number of Letters to be ſet down in forming all thoſe Products ſhould be $m \times (m - 1)$; wherefore the Number of all thoſe Products ſhould be $\frac{m \times (m-1)}{2}$, and this is the Value of $A$, or of the Coeficient of the third Term of the Formula ſought.

To find the Coeficient of the fourth Term, that is, the Number of different Products of three Letters $a\,b\,c$, $a\,b\,d$, $a\,c\,d$, $b\,c\,d$, &c. that a Number $m$ of Letters $a$, $b$, $c$, $d$, &c. will admit of, taken three by three, I obſerve firſt that this Number ſhould be the one third of the Letters ſet down in forming thoſe Products.

I obſerve next, that each of thoſe Letters ſhould be repeated the ſame Number of Times, and that this Number is the ſame as that which expreſſes how many different Products all the other Letters, taken two by two, will admit of; for it is manifeſt, that each Letter $a$ for Example, ſhould be joined to all the Products $b\,c$, $b\,d$, $c\,d$, &c. of the other Letters, taken two by two.

The number of Times therefore that each of the Letters $a$, $b$, $c$, $d$, &c. ſhould be repeated, is the ſame as that which expreſſes how many different Products a Number $m - 1$ of Letters $b$, $c$, $d$, &c. taken two by two, will admit of. Now we have ſeen that when the Number of Letters was $m$, the Number of their Products, when taken two by two, was expreſſed by the one Half of the Number $m$, multiplied by the Number $m - 1$. Wherefore, when the Number of Letters is $m - 1$, we ſhould take the one Half $\frac{m-1}{2}$ of this Number, and multiply it by $m - 2$, that is, $\frac{(m-1) \times (m-2)}{2}$ expreſſes the Number of Times that each of the Letters $a$, $b$, $c$, &c. ſhould be repeated in all the Products in Queſtion; and as the Number of thoſe Letters is $m$, conſequently, $\frac{m \times (m-1) \times (m-2)}{2}$ will be the Number of all the Letters ſet down; wherefore the Number ſought of the Products of three Letters $a\,b\,c$, $a\,b\,d$, &c. will be $\frac{m \times (m-1) \times (m-2)}{2 \times 3}$, and this is the Value of $B$, or of the Coeficient of the fourth Term.

To find the Coeficient $C$ of the fifth Term, that is, the Number of Products of four Letters, that a Number $m$ of Letters will admit of; I obſerve, that this Number ſhould be the one fourth of all the Letters ſet down in the Products; that each of thoſe Letters ſhould be repeated the ſame Number of Times, and ſhould be combined with all the Products of three Letters, that a Number $m - 1$ of Letters will admit of: and in fine, that thoſe Products of three Letters, that a Number $m - 1$ of Let-

ters will admit of, ſhould be expreſſed by $\frac{(m-1)\times(m-2)\times(m-3)}{2\times 3}$ for the ſame Reaſon that $\frac{m\times(m-1)\times(m-2)}{2\times 3}$ expreſſes the Number of Products of three Letters, that a Number $m$ Letters will admit of; hence the Number ſought of the Products of four Letters $abcd$, $abce$, &c. will be expreſſed by $\frac{m\times(m-1)\times(m-2)\times(m-3)}{2\times 3\times 4}$.

Forming after the ſame Manner the other Coeficients, and ſubſtituting in the foregoing Formula in the Room of $A$, $B$, $C$, $D$, $E$, &c. their Values thus found, there will reſult at length $x^m + m\,a\,x^{m-1} + m\times\frac{m-1}{2}\,a^2x^{m-2} + \frac{m\times(m-1)\times(m-2)}{2\times 3}\,a^3x^{m-3} + \frac{m\times(m-1)\times(m-2)\times(m-3)}{2\times 3\times 4}\,a^4x^{m-4} + \frac{m\times(m-1)\times(m-2)\times(m-3)\times(m-4)}{2\times 3\times 4\times 5}\,a^5x^{m-5}$, &c. for the Power $m$ of $x+a$.

In like Manner, the Value of $\overline{u+z}^m$ will be $u^m + m\,z\,u^{m-1} + \frac{m\times(m-1)}{2} + z^2u^{m-2}\ \frac{m\times(m-1)\times(m-2)}{2\times 3}\,z^3u^{m-3} + \frac{m\times(m-1)\times(m-2)\times(m-3)}{2\times 3\times 4}\,z^4u^{m-4}$, &c. General formula for raiſing $u+z$ to the power $m$.

As to the Value of $(u-z)^m$, it is manifeſt, in order to obtain it, it ſuffices to make $z$ negative in this Formula, which will transform it into $u^m - m\,z\,u^{m-1} + \frac{m\times(m-1)}{2}\,z^2u^{m-2} + \frac{m\times(m-1)\times(m-2)}{2\times 3}\,z^3u^{m-3} + \frac{m\times(m-1)\times(m-2)\times(m-3)}{2\times 3\times 4}\,z^4u^{m-4}$, &c.

XXII.

To raiſe any Binomial to a given Power by means of the foregoing Formula, we have no more to do but to ſubſtitute in the foregoing Value of $(u+z)^m$ in the Place of $u$ the firſt Term of the given Binomial, in the Place of $z$ the ſecond, and in the Place of $m$ the Exponent of the Power to which the propoſed Binomial is to be raiſed. Application of the foregoing formula to an example.

Let it be propoſed, for Example, to raiſe $3\,a\,c - 2\,b\,d$ to the fifth Power, I put $3\,a\,c = u$, $-2\,b\,d = z$, $5 = m$, and there reſults $u^m = (3\,a\,c)^5 = 243\,a^5c^5$, $m\,z\,u^{m-1} = 5\times -(2\,b\,d)\times(3\,a\,c)^4 = -810\,a^4\,b\,c^4d$, $\frac{m\times(m-1)}{2}\,z^2\,u^{m-2} = 10\times 4\,b\,b\,d\,d\times(3\,a\,c)^3 = 1080\,a^3\,b\,b\,c^3\,d\,d$, $\frac{m\times(m-1)\times(m-2)}{2\times 3}\,z^3\,u^{m-3} = 10\times -(2\,b\,d)^3\times(3\,a\,c)^2 = -720\,a^2\,b^3\,c^2\,d^3$.

$\frac{m \times (m-1) \times (m-2) \times (m-3)}{2 \times 3 \times 4} z^4 u^{m-4} = 5 \times -(2bd)^4 \times 3ac = 240 d b^4 c d^4$, $\frac{m \times (m-1) \times (m-2) \times (m-3) \times (m-4)}{2 \times 3 \times 4 \times 5} z^5 u^{m-5} = 1 \times -(2bd)^5 \times (3ac)^0 = -32 b^5 d^5$.

As to the other Terms, their Coeficients including among their Factors $m - 5$, which is equal to nothing when $m = 5$, they ſhould all vaniſh; whence the Value of $(3ac - 2bc)^5$ will be $243 a^5 c^5 - 810 a^4 b c^4 d + 1080 a^3 b^2 c^3 d^2 - 720 a^2 b^3 c^2 d^3 + 240 a b^4 c d^4 - 32 b^5 d^5$.

XXIII.

How the foregoing formula may be applied to quantities conſiſting of more than two terms.

If a Quantity conſiſting of more than two Terms is to be involved, it may be eaſily effected by the ſame Method. If it be a Trinomial, for Example, calling $u$ the firſt Term of this Trinomial, $z$ the Sum of the two other Terms, the Difficulty of raiſing a Trinomial to any Power will be reduced to that of raiſing a Binomial to the ſame Power, ſince each of the Terms $m u^{m-1} z$, $\frac{m \times (m-1)}{2} u^{m-2} z^2$, &c. does not include any Quantity to be involved more compoſed than the Binomial.

XXIV.

Example.

To ſhew by an Example how the foregoing Formula is to be employed for raiſing to any Power a Quantity conſiſting of more than two Terms, let it be propoſed to raiſe $a + 2b - c$ to the fourth Power. I put $m = 4$, $u = a$, $z = 2b - c$, and ſubſtituting thoſe Values in the Formula, there reſults $u^m = a^4$, $m z u^{m-1} = 4 \times (2b - c) \times a^3 = 8 a^3 b - 4 a^3 c$. $\frac{m \times (m-1)}{2} z^2 u^{m-2} = 6 \times (2b - c)^2 \times a^2 = 24 a^2 b^2 - 24 a^2 b c + 6 a^2 c^2$, $\frac{m \times (m-1) \times (m-2)}{2 \times 3} z^3 u^{m-3} = 4 \times (2b - c)^3 \times a = 32 a b^3 - 48 a b b c + 24 a b c c - 4 a c^3$, $\frac{m \times (m-1) \times (m-2) \times m-3)}{2 \times 3 \times 4} z^4 u^{m-4} = (2b - c)^4 = 16 b^4 - 4 \times (2b)^3 \times c + \frac{4 \times 3}{2} \times (2b)^2 \times c^2 - \frac{4 \times 3 \times 2}{2 \times 3} \times 2b \times c^3 + \frac{4 \times 3 \times 2 \times 1}{2 \times 3 \times 4} c^4 = 16 b^4 - 32 b^3 c + 24 b b c c - 8 b c^3 + c^4$, and conſequently, $(a + 2b - c)^4 = a^4 + 8 a^3 b - 4 a^3 c + 24 a^2 b^2 - 24 a^2 b c + 6 a^2 c^2 + 32 a b^3 - 48 a b^2 c + 24 a b c^2 - 4 a c^3 + 16 b^4 - 32 b^3 c + 24 b^2 c^2 - 8 b c^3 + c^4$.

XXV.

To find the Roots of Numbers by means of the foregoing Theorem, every fourth, fifth, and in general every $m^{th}$ Figure, beginning from the Units Place, is to be pointed, according as it is the Root of the fourth, fifth, or in general of the $m^{th}$ Power that is required; and if there be

any Decimals annexed to the Number, they are to be pointed after the ſame Manner, proceeding from the Units Place towards the right Hand, whereby it is eaſy to perceive, that the Number will be divided into ſo many Periods as there are Figures in the Root required. Then inquire which is the greateſt $m$ Power in the firſt Period to the Left of the given Number, and the Root of that Power will be the firſt Figure of the Root required, which call $u$, and ſubtract $u^m$ from this firſt Period. To find the ſecond Figure $z$ of the Root to the Remainder annex the firſt Figure of the ſecond Period, and this Number will contain the Term $m u^{m-1} z$ of the Binomial $u + z$ raiſed to the Power $m$, whence if $m u^{m-1} z$ be divided by $m u^{m-1}$, and after the Diviſion is performed, if what remains of the two firſt Periods does not exceed $\frac{m \times m-1}{2} u^{m-2} z^2 + \frac{m \cdot m-1 \cdot m-2}{2 \times 3} u^{m-3} z^3 + \frac{m \cdot m-1 \cdot m-2 \cdot m-3}{2 \times 3 \times 4} u^{m-4} z^4 +$, &c. the Quotient will be the ſecond Figure of the Root. The third Figure will be found by Means of the two firſt, as the ſecond was found by the firſt, and afterwards the fourth Figure (if there be a fourth Period) after the ſame Manner from the three firſt.

General rule for extracting the roots of numbers derived from the foregoing theorem.

XXVI.

Let the fifth Root of 6436343 be required, I firſt point it 64,36343, the Formula of the fifth Power is $u^5 + 5 u^4 z + 10 u^3 z^2 + 10 u^2 z^3 + 5 u z^4 + z^5$; wherefore 64 is either equal to, or contains $u^5$, it is not equal to it, ſince no Root raiſed to the fifth Power will give 64; conſequently it contains it, but the greateſt fifth Power in 64 is 32, whoſe fifth Root is 2; wherefore $2 = u$. I ſubtract 32 from 64, and to the Remainder 32 I annex 3, the firſt Figure of the ſecond Period, and there reſults 323, now $u^4 = 16$, and $5 u^4 = 80$; I therefore divide 323 by 80, and the Quotient is 4, leaving a Remainder 3, which annexed to the remaining Figures in the propoſed Number, there reſults 36343, which ſhould not exceed $10 u^3 z^2 + 10 u^2 z^3 + 5 u z^4 + z^5$, but $10 u^3 z^2$ is equal to 1280 Thouſands, $10 u^2 z^3$ to 2560 Hundreds, which added to the 1280 Thouſands already found, makes 15360 Hundreds, and as this Number exceeds 363 Hundreds that remain in the propoſed Power, the ſecond Part of the Root 4 has not been rightly determined: Let the ſecond Figure of the Root be 3, then $3 \times 80 = 240$, and $323 - 240 = 83$, wherefore there will remain 836343 which ſhould not exceed $10 u^3 z^2 + 10 u^2 z^3 + 5 u z^4 + z^5$; now $10 u^3 z^2 = 720$ Thouſands, and $10 u^2 z^3 = 1080$ Hundreds, which added to 720 Thouſands, makes 8280 Hundreds, and $5 u z^4 = 810$ Tens, which added to 8280 Hundreds, makes 83610 Tens; laſtly, $z^5 = 243$, which added to the 83610 Tens, makes 836343; wherefore 3 is the ſecond Figure of the Root required.

Application of this method to an example.

XXVII.

To approximate to the $n$ Root of any Number $N$, let $r$ denote the neareſt leſs Root in Integers, and $r + z$ the true Root: Then will

$$N = r^n + nr^{n-1}z + n \cdot \frac{n-1}{2} \cdot r^{n-2}z^2 + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} r^{n-3}z^3$$
$$+ n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} r^{n-4}z^4, \text{ &c. or } - \frac{m}{nr^{n-1}} + z$$
$$+ \frac{n-1}{2} \cdot \frac{1}{r} \cdot z^2 + \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{1}{r^2} \cdot z^3$$
$$+ \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot \frac{1}{r^3} \cdot z^4, \text{ &c.} = 0.$$

General method of approximating to the roots of numbers.

(putting $N - r^n = m$). Multiplying this Equation by $1 + Az$, there will reſult,

$$- \frac{m}{nr^{n-1}} + \left(1 - \frac{mA}{nr^{n-1}}\right) z + \left(\frac{n-1}{2} \cdot \frac{1}{r} + A\right) z^2$$
$$+ \left(\frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{1}{r^2} + \frac{n-1}{2} \cdot \frac{A}{r}\right) z^3$$
$$+ \left(\frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot \frac{1}{r^3} + \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{A}{r^2}\right) z^4, \text{ &c.} = 0$$

in which the third Term will be deſtroyed by determining $A$ by Means of the Equation $\frac{n-1}{2} \cdot \frac{1}{r} + A = 0$; from which Equation $A$ is found $= - \frac{n-1}{2} \cdot \frac{1}{r}$, and by ſubſtituting this Value above, we have

$$- \frac{m}{nr^{n-1}} + \left(1 + \frac{n-1 \cdot m}{2nr^n}\right) z - \frac{n+1}{4} \cdot \frac{n-1}{3} \cdot \frac{1}{r^2} \cdot z^3$$
$$- \frac{n+1 \cdot n-1 \cdot n-2}{4 \times 3 \times 2} \cdot \frac{z^4}{r^3} - \frac{n+1 \cdot n-1 \cdot n-2 \cdot n-3}{5 \times 4 \times 3 \times 2} \cdot \frac{z^5}{r^4}, \text{ &c.} = 0$$

the Terms of which Series decreaſes very ſwiftly, neglecting, therefore, all the Terms after the ſecond as exceeding ſmall, it will be reduced to

$$- \frac{m}{nr^{n-1}} + \left(1 + \frac{n-1 \cdot m}{2nr^n}\right) z = 0, \text{ whence } z = \frac{rm}{nr^n + \frac{n-1}{2} \cdot m}.$$

To obtain a more approximate Value of $z$, it is ſufficient to multiply

$$- \frac{m}{nr^{n-1}} + z + \frac{n-1}{2} \cdot \frac{1}{r} \cdot z^2 + \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{1}{r^2} z^3$$
$$+ \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot \frac{1}{r^3} \cdot z^4, \text{ &c.} = 0, \text{ by } 1 + Az + Bz^2,$$

whence $- \frac{m}{nr^{n-1}} + \left(1 - \frac{mA}{nr^{n-1}}\right) z + \left(\frac{n-1}{2r} + A - \frac{mB}{nr^{n-1}}\right) z^2$

$$+ \left(\frac{n-1 \cdot n-2}{2 \times 3 \times r^2} + \frac{n-1 \cdot A}{2 \times r} + B\right) z^3 + \left(\frac{n-1 \cdot n-2 \cdot n-3}{2 \times 3 \times 4 \times r^3}\right.$$
$$\left. + \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{A}{r^2} + \frac{n-1}{2} \cdot \frac{B}{r}\right) z^4, \text{ &c.} = 0.$$

In which Equation the third and fourth Terms will be deſtroyed, by determining $A$ and $B$ by Means of the Equations

$$\frac{n-1}{2\times r}+A-\frac{mB}{nr^{n-1}},=0,\quad \frac{n-1}{2}\cdot\frac{n-2}{3\times r^2}\cdot+\frac{n-1}{2}\cdot\frac{A}{r}+B=0$$

Let the latter of thoſe Equations be multiplied by $\frac{m}{nr^{n-1}}$, and then add the two Products together, ſo ſhall

$$\frac{n-1}{2}\cdot\frac{1}{r}+A+\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{m}{nr^{n+1}}+\frac{n-1}{2}\cdot\frac{mA}{nr^n}=0.$$

whence is deduced $\dfrac{\frac{n-1}{2}\cdot\frac{1}{r}+\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{m}{nr^{n+1}}}{1+\frac{n-1}{2}\cdot\frac{m}{nr^n}}=-A.$

To render this Value of $-A$ more ſimple, I obſerve, that if two Quantities be decreaſed by two other ſmall Quantities nearly in the ſame Ratio with the two firſt, that their Difference will ſtill be in the ſame Ratio with the two firſt Quantities very near; wherefore, ſeeing the Numerator of this Fraction is in Proportion to the Denominator nearly as $\frac{n-1}{2}\cdot\frac{1}{r}:1$, or as $\frac{n-1\cdot n-2}{2\times 3}\cdot\frac{m}{nr^{n+1}}:\frac{n-2}{3}\cdot\frac{m}{nr^n}$, let $\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{m}{nr^{n+1}}$ be therefore taken from the Numerator, and $\frac{n-2}{3}\cdot\frac{m}{nr^n}$, from the Denominator; by this Means the Fraction itſelf will be reduced to $\dfrac{\frac{1}{2}\cdot(n-1)\cdot\frac{1}{r}}{1+(n+1)\cdot\frac{1}{6}\cdot\frac{m}{nr^n}}$

or to $\dfrac{\frac{1}{2}\cdot(n-1)}{r+(n+1)\cdot\frac{1}{6}\cdot\frac{m}{nr^{n-1}}}$, whence putting $\frac{N-r^n}{nr^n}=p$ we will have $z=\dfrac{p+(n+1)\times\frac{1}{6}p^2}{1+(2n-1)\cdot\frac{1}{3}p}$ for an Approximation of the third Degree.

If more Terms of the Series $1+Az+Bz^2+Cz^3$, &c. are taken the Root will be found exacter in Proportion; for Example, if four Terms of this Series were employed, there would reſult for an Approximation of the fourth Degree $-A=\dfrac{\frac{1}{2}\times(n-1)+(n-1\times(2n-1)\times\frac{1}{12}p}{1+\frac{1}{2}np}$

and $z=\dfrac{p+\frac{1}{2}np^2}{1+\frac{2n-1}{2}p+\frac{(2n-1)\times(n-1)}{12}p^2}$.

But both those Expressions, in Cases where $p$ is a proper Fraction, will be better adapted to Practice by making $\frac{n r^n}{N - n r^n} = v = \frac{1}{p}$, and substituting $\frac{1}{v}$ for $p$ its Equal; whence after proper Reductions $x = r + \frac{r}{v} \times \frac{6v+n+1}{6v+4n-2}$ nearly; or $x = r + \frac{r \times (2v+n)}{v \times (2v+2n-1) + \frac{1}{6} \times (2n-1) \times (n-1)}$

XXVIII.

Application of the foregoing approximations to an example.

To shew the Use and great Exactness of those Approximations by an Example, let the Equation $x^3 = 500$ be given, then assuming $r = 8$ we have $z = -\frac{32}{508}$, and consequently $r + z = 8 - \frac{32}{508} = 7{,}937$ by the first Approximation.

By the second we have $v = \frac{n r^n}{N - r^n} = \frac{3 \times 512}{-12}$, hence by the second Approximation $x = 8 - \frac{1}{16} \times \frac{-768+4}{-768+10} = 8 - \frac{191}{3032}$ $= 7{,}93700527$ nearly. And by the third $x = 8 - \frac{8 \times 253}{128 \times 251 + \frac{5}{3}}$ $= 8 - \frac{6072}{96389} = 7{,}9370052599$.

When a near Value of $z$ is determined, then adding it to $r$, substitute the Aggregate in the Place of $N$ in the Formulas, and you will by a new Operation obtain a more correct Value of the Root required; and by thus proceeding you may arrive at any Degree of Exactness.

XXIX

Use of the foregoing rules in the solution of problems. First example.

To shew the Use of the foregoing Rules in the Solution of Problems, here follow some Examples.

*Suppose that out of a Cask holding* c *Gallons of Wine when full, a certain Quantity* x *was drawn and the Cask filled up with Water, and that the same Quantity of the Mixture was afterwards drawn, and supplied by Water* t *several Times, and then it appeared that besides Water there were but* r *Gallons of Wine left in the Cask: How much Wine was drawn out each Time?*

$c - x$ expresses the Quantity of Wine left after the first Drawing; but $c : c - x = x : \frac{(c-x) \times x}{c}$ the Quantity of Wine drawn out at second Drawing, and $c - x - \frac{(c-x) \cdot x}{c} = \frac{(c-x)^2}{c}$ the Quantity of Wine left after the third Drawing:

Also $c : \frac{(c-x)^2}{c} = x : \frac{(c-x)^2 \cdot x}{c^2}$ the Quantity of Wine drawn out at the fourth Drawing, and $\frac{(c-x)^2}{c} - \frac{(c-x)^2 \cdot x}{cc} = \frac{(c-x)^3}{cc}$

equal the Quantity of Wine left after the fourth Drawing; then $c : \frac{(c-x)^{t-1}}{c^{t-2}} = x : \frac{(c-x)^{t-1} \cdot x}{c^{t-1}}$ equal the Quantity of Wine drawn out at the $t$ Drawing, and $\frac{(c-x)^{t-1}}{c^{t-2}} - \frac{(c-x)^{t-1} \cdot x}{c^{t-1}} = \frac{(c-x)^t}{c^{t-1}}$ the Quantity of Wine left after the $t$ Drawing. Now $\frac{(c-x)^t}{c^{t-1}} = r$ by the Queſtion, or $(c-x)^t = rc^{t-1}$ and $c-x = rc^{t-1}\Big]^{\frac{1}{t}}$ and $c - rc^{t-1}\Big]^{\frac{1}{t}} = x$.

To apply this general Solution to an Example, let the Contents of the Caſk be 81 Gallons, the Wine remaining 16 Gallons, the Number of Times Liquor was drawn out 4, then $x = 88 - \sqrt[4]{8503056} = 27$ Gallons.

XXX.

*If every Century the Number of Men were doubled, what would be the annual Increaſe?* Second example.

Let the primitive Number of Men be expreſſed by $n$, and let their Number be increaſed annually by their $\frac{1}{x}$ Part.

Their Number at the End of the firſt Year will be expreſſed by $n + \frac{n}{x}$, at the End of the ſecond Year by $\frac{n+nx}{x} + \frac{n+nx}{x^2}$, or by $\left(\frac{1+x}{x}\right)^2 n$; at the End of the third Year by $\left(\frac{1+x}{x}\right)^3 n$; and by $\left(\frac{1+x}{x}\right)^{100} n$ at the End of the 100th Year. Now by the Queſtion $\left(\frac{1+x}{x}\right)^{100} n = 2n$, wherefore $\frac{1+x}{x} = 2^{\frac{1}{100}}$.

Whence $\frac{1+x}{x} = \frac{10070240}{10000000}$, wherefore $x = \frac{10000000}{70240} = 144$ nearly, which ſhews that the Number of Men ſhould be annually increaſed by their $\frac{1}{144}$ Part, in order that every Century their Number ſhould be doubled: It is therefore no way ſurprizing that the World ſhould have been peopled by one Pair.

XXXI.

After the Analyſts had found the general Theorem for the Involution of Binomials, they ſoon perceived that it might be extended to other Powers beſides thoſe whoſe Exponents are whole and poſitive Numbers; having found, for Example, that $a^{\frac{1}{n}}$ might be ſubſtituted for $\sqrt[n]{a}$; they concluded, without Doubt, that to extract the Root $n$ of any compound

Quantity represented by $u + z$; it sufficed to suppose $m = \frac{1}{n}$ in the foregoing Formula, or what amounted to the same thing that $u^{\frac{1}{n}} + \frac{1}{n} u^{\frac{1}{n}-1} z + \frac{\frac{1}{n} \times \frac{1}{n} - 1}{2} u^{\frac{1}{n}-2} z^2 + \&c. = (u+z)^{\frac{1}{n}}$, or the Root $n$ of $u + z$.

In like Manner, that $\frac{1}{(u+z)^n}$, or $(u+z)^{-n} = u^{-n} - n u^{-n-1} z - \frac{n \times -n-1}{2} u^{-n-2} z^2 - \&c.$ In one Word the Order and Generality that the Analysts always found in the analitick Operations, easily suggested to them, that though the foregoing Formula had been investigated only in the Supposition that $m$ was a whole and positive Number, however that it might be applied to all the other Values of $m$.

Let it first be proposed to shew that the Formula in Question serves to raise a Quantity to a Power whose Exponent is a Fraction, or what amounts to the same thing, let it be proposed to prove that

$$(u+z)^{\frac{r}{n}} = u^{\frac{r}{n}} + \frac{r}{n} u^{\frac{r}{n}-1} z + \frac{\frac{r}{n} \times \frac{r}{n} - 1}{2} u^{\frac{r}{n}-2} z^2 +, \&c.$$

which by dividing the two Members by $u^{\frac{r}{n}}$ and putting $\frac{z}{u} = p$ becomes

$$(1+p)^{\frac{r}{n}} = 1 + \frac{r}{n} p + \frac{\frac{r}{n} \times \frac{r}{n} - 1}{2} p^2 + \frac{\frac{r}{n} \times \frac{r}{n} - 1 \times \frac{r}{n} - 2}{2 \times 3} p^3 +, \&c.$$

To prove this Equation it suffices to shew that its two Members being raised to the same Power $n$, there will result equal Quantities, that is, putting

$$s = \frac{r}{n} p + \frac{\frac{r}{n} \times \frac{r}{n} - 1}{2} p^2 + \frac{\frac{r}{n} \times \frac{r}{n} - 1 \times \frac{r}{n} - 2}{2 \times 3} p^3 +, \&c.$$

$(1+p)^r = (1+s)^n$.

Now as $r$ and $n$ are two whole Numbers, the Members of the foregoing Equation may be raised to the Powers indicated by their Exponents, by Means of the general Formula $(u+z)^m$. Whence

$$1 + rp + \frac{r(r-1)}{2} p^2 + \frac{r(r-1)(r-2)}{2 \times 3} p^3 + \frac{r(r-1)(r-2)(r-3)}{2 \times 3 \times 4} p^4 +, \&c.$$
$$= 1 + ns + \frac{n(n-1)}{2} s^2 + \frac{n(n-1)(n-2)}{2 \times 3} s^3 + \frac{n(n-1)(n-2)(n-3)}{2 \times 3 \times 4} s^4 + \&c$$

The Problem being reduced to prove this Equation, we must find by Means of the Value of $s$ those of $s^2, s^3, s^4$, &c. and then multiply the Va-

lue of $s$ by $n$, that of $s^2$ by $\frac{n(n-1)}{2}$, that of $s^3$ by $\frac{n(n-1)(n-2)}{2\times 3}$, that of $s^4$ by $\frac{n(n-1)(n-2)(n-3)}{2\times 3\times 4}$, &c. to obtain the Values of all the Terms of the ſecond Member of the foregoing Equation, thoſe Values being found, and ſet down under one another, there reſults

$$1+ns = 1+n.\frac{r}{n}p+n.\frac{\frac{r}{n}\cdot\frac{r}{n}-1}{2}p^2+n.\frac{\frac{r}{n}\cdot\frac{r}{n}-1\cdot\frac{r}{n}-2}{2\times 3}p^3+n.\frac{\frac{r}{n}\cdot\frac{r}{n}-1\cdot\frac{r}{n}-2\cdot\frac{r}{n}-3}{2\times 3\times 4}p^4+,\&c.$$

$$+\frac{n.n-1}{2}s^2 = +\frac{n.n-1}{2}\cdot\frac{r^2}{n^2}p^2+\frac{n.n-1}{2}\cdot\frac{r^2}{n^2}\cdot\frac{r}{n}-1\cdot p^3+\frac{n.n-1}{2}\cdot\frac{r^2}{n^2}\cdot\frac{r}{n}-1\cdot\frac{\frac{7r}{n}-11}{12}p^4+,\&c.$$

$$+\frac{n.n-1.n-2}{2\times 3}s^3 = +\frac{n.n-1.n-2}{2\times 3}\cdot\frac{r^3}{n^3}p^3+\frac{n.n-1.n-2}{2\times 3}\cdot\frac{3r^3}{2n^3}\cdot\frac{r}{n}-1\,p^4+,\&c.$$

$$+\frac{n.n-1.n-2.n-3}{2\times 3\times 4}s^4 = +\frac{n.n-1.n-2.n-3}{2\times 3\times 4}\cdot\frac{r^4}{n^4}p^4+,\&c.$$

$$+,\&c. \qquad +,\&c.$$

Reducing the Terms of the ſecond Member of this Equation, there reſults, $1+ns+\frac{n.n-1}{2}s^2+\frac{n.n-1.n-2}{2\times 3}s^3+\frac{n.n-1.n-2.n-3}{2\times 3\times 4}s^4+,\&c.$

$$= 1+rp+\frac{r.r-1}{2}p^2+\frac{r.r-1.r-2}{2\times 3}p^3+\frac{r.r-1.r-2.r-3}{2\times 3\times 4}p^4+,\&c.$$

which is preciſely the foregoing Equation. Wherefore the general Theorem found for the Involution of Binomials will ſerve alſo for their Evolution.

The general formula found for the involution of binomials, ſerves alſo for their evolution.

XXXI.

Let it now be propoſed to prove that the ſame Formula ſerves to raiſe Quantities to negative Powers, either Integral or Fractional, or that

$$(u+z)^{-\frac{r}{n}} = u^{-\frac{r}{n}} - \frac{r}{n}u^{-\frac{r}{n}-1}z+\frac{\frac{r}{n}\cdot\frac{r}{n}+1}{2}u^{-\frac{r}{n}-2}z^2+\frac{\frac{r}{n}\cdot\frac{r}{n}+1\cdot\frac{r}{n}+2}{2\times 3}u^{-\frac{r}{n}-3}z^3+,\&c.$$

which Equation by multiplying the two Members by $u^{\frac{r}{n}}$, and putting $p=\frac{z}{u}$ is reduced to

$$(1+p)^{-\frac{r}{n}} = 1-\frac{r}{n}p+\frac{\frac{r}{n}\cdot\frac{r}{n}+1}{2}p^2+\frac{\frac{r}{n}\cdot\frac{r}{n}+1\cdot\frac{r}{n}+2}{2\times 3}p^3+\frac{\frac{r}{n}\cdot\frac{r}{n}+1\cdot\frac{r}{n}+2+\frac{r}{n}+3}{2\times 3\times 4}p^4+,\&c.$$

It is manifeſt that inſtead of the Quantity $(1+p)^{-\frac{r}{n}}$ we may ſet

down $\frac{1}{(1+p)^{\frac{r}{n}}}$, consequently the foregoing Equation is transformed into

$$\frac{1}{(1+p)^{\frac{r}{n}}} = 1 - \frac{r}{n}p + \frac{\frac{r}{n}\cdot\frac{r}{n}+1}{2}p^2 - \frac{\frac{r}{n}\cdot\frac{r}{n}+1\cdot\frac{r}{n}+2}{2\times 3}p^3 +, \&c.$$

But as we can substitute for $(1+p)^{\frac{r}{n}}$ its Value deduced from the Formula of the foregoing Article, it is manifest that it suffices to prove that

$$\frac{1}{1+\frac{r}{n}p+\frac{\frac{r}{n}\cdot\frac{r}{n}-1}{2}p^2+\frac{\frac{r}{n}\cdot\frac{r}{n}-1\cdot\frac{r}{n}-2}{2\times 3}p^3+\frac{\frac{r}{n}\cdot\frac{r}{n}-1\cdot\frac{r}{n}-2\cdot\frac{r}{n}-3}{2\times 3\times 4}p^4+, \&c.}$$

$$= 1 - \frac{r}{n}p + \frac{\frac{r}{n}\cdot\frac{r}{n}+1}{2}p^2 - \frac{\frac{r}{n}\cdot\frac{r}{n}+1\cdot\frac{r}{n}+2}{2\times 3}p^3 + \frac{\frac{r}{n}\cdot\frac{r}{n}+1\cdot\frac{r}{n}+2\cdot\frac{r}{n}+3}{2\times 3\times 4}p^4 -, \&c.$$

Now to prove this Equation, it suffices to multiply the second Member by the Denominator of the first, and to make it appear that the Product that results is Unity, the Numerator of the first Member: which happens in effect; for performing the Multiplication, there results

$$\begin{aligned}
&1 - \frac{r}{n}p + \frac{\frac{r}{n}\cdot\frac{r}{n}+1}{2}p^2 - \frac{\frac{r}{n}\cdot\frac{r}{n}+1\cdot\frac{r}{n}+2}{2\times 3}p^3 + \frac{\frac{r}{n}\cdot\frac{r}{n}+1\cdot\frac{r}{n}+2\cdot\frac{r}{n}+3}{2\times 3\times 4}p^4 -, \&c.\\
&+\frac{r}{n}p - \frac{r}{n}\cdot\frac{r}{n}p^2 + \frac{\frac{r}{n}\cdot\frac{r}{n}\cdot\frac{r}{n}+2}{2}p^3 - \frac{\frac{r}{n}\cdot\frac{r}{n}\cdot\frac{r}{n}\cdot\frac{r}{n}+1\cdot\frac{r}{n}+2}{2\times 3\times 4}p^4 +, \&c.\\
&+\frac{\frac{r}{n}\cdot\frac{r}{n}-1}{2}p^2 - \frac{\frac{r}{n}\cdot\frac{r}{n}\cdot\frac{r}{n}-1}{2}p^3 + \frac{\frac{r}{n}\cdot\frac{r}{n}\cdot\frac{r}{n}+1\cdot\frac{r}{n}-1}{2\times 3}p^4 -, \&c.\\
&+\frac{\frac{r}{n}\cdot\frac{r}{n}-1\cdot\frac{r}{n}-2}{2\times 3}p^3 - \frac{\frac{r}{n}\cdot\frac{r}{n}\cdot\frac{r}{n}-1\cdot\frac{r}{n}-2}{2\times 3}p^4 +, \&c.\\
&+\frac{\frac{r}{n}\cdot\frac{r}{n}-1\cdot\frac{r}{n}-2\cdot\frac{r}{n}-3}{2\times 3\times 4}p^4 -, \&c.\\
&+, \&c.
\end{aligned}$$

whose first Term, which is Unity, is the only one that remains after the the Reduction.

XXXIII.

Since to extract any Root of a given Quantity, is the same Thing as to raise that Quantity to a Power, whose Exponent is a Fraction that has its

Denominator equal to the Number that expreſſes what Kind of Root is to be extracted, it is plain that the Root of any Quantity may be found by the foregoing Formula, by putting $m = \frac{1}{2}$ or $\frac{1}{3}$ or $\frac{1}{4}$, &c. according as it is the ſquare Root, cube Root, or the Root of the fourth Power, &c. that is required; and if the propoſed Quantity is an exact Square, or an exact Cube, &c. the Series will ſtop of itſelf, the Numerator of one of the Terms, in this Caſe vaniſhing; if not, it may be carried on to any Number of Terms, as will appear by the following Examples.

Application of the foregoing formula for finding the roots of perfect powers of compound quantities.

Let it be propoſed to extract the ſquare Root of $1 + 2b + b^2$, putting $u = 1$, $z = 2b + b^2$, and $m = \frac{1}{2}$ in the foregoing Formula, I find the firſt Term $u^m = 1$, the ſecond $m u^{m-1} z = b + \frac{1}{2} b^2$, the third $\frac{m \times (m-1)}{2} u^{m-2} z^2 = -\frac{1}{2} b^2 - \frac{1}{2} b^3 - \frac{1}{8} b^4$; the 4th. Term. $\frac{m \times (m-1) \times (m-2)}{2 \times 3} u^{m-3} z^3 = \frac{1}{2} b^3 + \frac{3}{4} b^4 + \frac{3}{8} b^5 + \frac{1}{16} b^6$ the 5th. $\frac{m \times (m-1) \times (m-2) \times (m-3)}{2 \times 3 \times 4} u^{m-4} z^4 = -\frac{5}{8} b^4 - \frac{5}{4} b^5 - \frac{15}{16} b^6 - \frac{15}{16} b^7 - \frac{5}{128} b^8$. The 6th., &c. and the Sum of all thoſe Terms will be the Root required.

Adding all thoſe Terms, I obſerve that the firſt 1 remains entirely, that the 2d. $b + \frac{1}{2} b^2$ is reduced to $b$, becauſe the Part $\frac{1}{2} b^2$ of this ſecond Term is deſtroyed by the ſame Quantity with a negative Sign in the third Term $-\frac{1}{2} b^2 - \frac{1}{2} b^3 - \frac{1}{8} b^4$; that what remains of the third Term $-\frac{1}{2} b^3 - \frac{1}{8} b^4$ after $\frac{1}{2} b^2$ is deſtroyed, ſhould be likewiſe entirely deſtroyed by the fourth $\frac{1}{2} b^3 + \frac{3}{4} b^4 + \frac{3}{8} b^5 + \frac{1}{16} b^6$, of which after the Reduction there only remains $\frac{5}{8} b^4 + \frac{3}{8} b^5 + \frac{1}{16} b^6$, this Remainder of the fourth Term is deſtroyed in like Manner by the fifth Term, and continuing the Reduction further on, it will appear that the Whole vaniſhes except $1 + b$, which is the Root required.

XXXIV.

Application of the foregoing formula for finding

Let the Root of the 5th. Power of the Quantity $a + b$ be required, ſubſtituting the greateſt of the two Parts of this Quantity, which I ſuppoſe to be $a$ in the Room of $u$, the leaſt Quantity $b$ in the Room of $z$, and $\frac{1}{5}$ in the Room of $m$, there will reſult for the Root required

ing the roots of imperfect powers of compound quantities.

$$a^{\frac{1}{5}} + \frac{1}{5}a^{-\frac{4}{5}}b - \frac{1\times 4}{2\times 25}a^{-\frac{9}{5}}b^2 + \frac{1\times 4\times 9}{2\times 3\times 125}a^{-\frac{14}{5}}b^3$$

$$- \frac{1\times 4\times 9\times 14}{2\times 3\times 4\times 625}a^{-\frac{19}{5}}b^4 + \frac{1\times 4\times 9\times 14\times 19}{2\times 3\times 4\times 5\times 3125}a^{-\frac{24}{5}}b^5 -, \&c.$$

$$\text{or } a^{\frac{1}{5}}\times\left(1 + \frac{b}{5a} - \frac{2bb}{25aa} + \frac{6b^3}{125a^3} - \frac{21b^4}{625a^4} +, \&c.\right)$$

Though it be requiſite to carry on this Series to an infinite Number of Terms, in order that it may expreſs exactly the Root required, however, as this Series will converge ſooner, the greater $a$ is in Reſpect of $b$, by taking a Number of its Terms we ſhall approximate to the Root to any aſſigned Degree of Exactneſs.

Let, for Example, $b = \frac{1}{10}a$ the ſix firſt Terms of the Series will be

$$a^{\frac{1}{5}}\times\left(1 + \frac{1}{50} - \frac{2}{1250} + \frac{3}{62500} - \frac{21}{6210000} + \frac{399}{1561500000}\right),$$

which converges very ſwiftly.

XXXV.

The foregoing formula applied to raiſing quantities to negative powers.

Let it be propoſed to approximate to the Quotient of $\frac{aa}{x \pm b}$, I bring the Quantity $x \pm b$ from the Denominator to the Numerator, by changing the Sign of the Exponent 1 into its oppoſite $-1$, ſo that the Expreſſion ſtands thus, $aa \times (x \pm b)^{-1}$: I then reduce $(x \pm b)^{-1}$ into a Series, by ſubſtituting the greateſt of the two Parts $x$ in the Room of $z$, and the leaſt $\pm b$ in the Room of $z$, and $-1$ in the Room of $m$, and I find

$$aa \times (x \pm b)^{-1} = \frac{aa}{x} \mp \frac{aab}{x^2} + \frac{aab^2}{x^3} \mp \frac{a^2b^3}{x^4} + \frac{a^2b^4}{x^5} + \&c.$$

XXXVI.

Difficulty that occurs when the two terms of the denominator are equal.

If in the propoſed Quantity $\frac{aa}{b \pm x}$, $b$ was $= x$, then we will have $1^0.\ \frac{aa}{b-b} = \frac{aa}{b}\times(1+1+1+1+1+1+1+, \&c.)$, whoſe Value is infinite, which is no more than what we knew already, ſince $\frac{aa}{b-b} = \frac{aa}{0} = \infty$. The Sign $\infty$ being employed by the Analyſts to expreſs an infinite Number.

$$2^0.\ \frac{aa}{b+b} = \frac{aa}{b}\times(1-1+1-, \&c.) = \frac{aa}{b}\times(0+0+, \&c.$$

whence $\frac{aa}{b+b} = 0$, which is not true, conſequently a Fraction whoſe

Denominator consists of two equal Terms cannot in that State be reduced into a Series: To remedy this Inconveniency, the Analysts divide the Sum of the two Parts of the Denominator into other unequal Parts: For Example, in the proposed Fraction $\frac{a\,a}{b+b}$ let $c = 2\,b + x$, and the Fraction will be transformed into $\frac{a\,a}{c-x}$, whose Denominator being $= 2\,b$, the Series arising from it will be equal $\frac{a\,a}{b+b}$.

XXXVII.

The Utility of the foregoing Formula is not confined to the finding by Approximation all Sorts of fractional or negative Powers; it is of infinite Use for reducing complicated Radicals: thus, if it was proposed to reduce $\frac{\sqrt{[c^4-(c\,c+r\,r)\,x^2]}}{\sqrt{(c^4-c\,c\,x^2)}}$ to an infinite Series, I bring the Quantity $\sqrt{(c^4-c\,c\,x^2)}$ from the Denominator to the Numerator, by changing the Sign of the Exponent $\frac{1}{2}$ into its opposite, so that the Expression stands thus, $[c^4-(c\,c-r\,r)\,x^2]^{\frac{1}{2}} \times (c^4-c\,c\,x^2)^{-\frac{1}{2}}$, which by substituting for Brevity sake, $a$ for $c\,c$ and $b$ for $-c\,c+r\,r$ will become $(a^2+b\,x^2)^{\frac{1}{2}} \times (a^2-a\,x^2)^{-\frac{1}{2}}$, reducing those two Factors into infinite Series, and afterwards multiplying them into one another, there results, $1+\frac{a+b}{2\,a^2}\,x^2+\frac{3\,a^2+2\,a\,b-b^2}{8\,a^4}\,x^4+\frac{5\,a^3+3\,a^2\,b-a\,b^2+b^3}{16\,a^6}\,x^6+\frac{35\,a^4+20\,a^3\,b-6\,a^2\,b^2+4\,a\,b^3-5\,b^4}{128\,a^8}\,x^8$, &c. for the Value of $\frac{\sqrt{(a^2+b\,x^2)}}{\sqrt{(a^2-a\,x^2)}}$.

All kinds of quantities may be reduced into series by the foregoing formula.

XXXVIII.

We have seen (Chap. II. Art. XXXIX, XL, XLI.) how the Analysts perform upon Radicals of the second Degree the Operations of Addition, Subtraction, Multiplication, and Division; those Operations being equally necessary for radical Quantities of higher Degrees, we shall proceed to explain what those new Radicals require.

Addition & subtraction of all sorts of radical quantities.

As to Addition and Subtraction, they require no more than what has been said already of those Operations upon Radicals of the second Degree: it suffices to reduce each Radical to its most simple Expression, and to add and subtract them as commensurable Quantities.

Let it be proposed, for Example, to subtract $\sqrt[3]{(b^4+2\,a\,b^3)}$ from $\sqrt[3]{(8\,a^3\,b+16\,a^4)}$; I transform the first Quantity into $b\,\sqrt[3]{(b+2\,a)}$, and the second into $2\,a\,\sqrt[3]{(b+2\,a)}$, and then subtracting one from the

the other, the Difference is $(2a - b)\sqrt[3]{(b + 2a)}$: In like Manner, $3a\sqrt[4]{(16b^8 + 32b^4a^4)}$ being added to $4b\sqrt[4]{(a^4b^4 + 2a^8)}$ their Sum will be $10ab\sqrt[4]{(b^4 + 2a^4)}$.

XXXIX.

Multiplication and Division of radical quantities that have the same exponent.

As to Multiplication and Division, if the radical Quantities have the same Exponent, the Method is the same as for Radicals of the second Degree; it suffices to operate upon the Quantities to which the radical Sign is prefixed, and set the same radical Sign over their Product or Quotient.

Thus, $\sqrt[3]{5ayy} \times \sqrt[3]{7ayz} = \sqrt[3]{35aay^3z}$, or $y\sqrt[3]{35a^2z}$

$\sqrt[5]{\frac{9a^2b^3}{4}} \times \sqrt[5]{27a^3b^6} = \sqrt[5]{\frac{243a^5b^9}{4}} = 3ab\sqrt[5]{\frac{b^4}{4}}$

$\sqrt[3]{(a^2b^2+b^4)}$ divided by $\sqrt[3]{\frac{a^2-b^2}{8b}}$ gives for Quotient $\sqrt[3]{\frac{8a^2b^3+8b^5}{a^2-b^2}}$

$= 2b\sqrt[3]{\frac{a^2+b^2}{a^2-b^2}}$, $\frac{\sqrt[3]{(a^2x^3 - x^5)}}{\sqrt[3]{(ab^4 + b^4x)}} = \frac{x}{b}\sqrt[3]{\frac{a-x}{b}}$.

XL.

To perform those operations upon radical quantities of different exponents, they are first to be reduced to the same exponent.

But to perform those Operations on radical Quantities of different Exponents, they must first be reduced to others of equivalent Value that shall have the same radical Sign.

Let, for Example, $\sqrt[3]{ab}$ and $\sqrt[5]{abbb}$ be proposed to be reduced to the same radical Sign, I raise $ab$ to the fifth Power, and set down $\sqrt[15]{}$ in the Place of $\sqrt[3]{}$ and there results $\sqrt[15]{a^5b^5} = \sqrt[3]{ab}$, I likewise raise $ab^2$ to the third Power, and set down $\sqrt[15]{}$ instead of $\sqrt[5]{}$, whence $\sqrt[5]{ab^2} = \sqrt[15]{a^3b^6}$.

Now if it was proposed to multiply $\sqrt[3]{ab}$ by $\sqrt[5]{abbb}$ the Product would be $\sqrt[15]{a^8b^{11}}$, if the first was to be divided by the second the Quotient would be $\sqrt[15]{\frac{a^5b^5}{a^3b^6}} = \sqrt[15]{\frac{a^2}{b}}$.

In like Manner the Product of $\sqrt[3]{a^2b^4}$ into $\sqrt{a^4b^5}$ would be $\sqrt[6]{a^{16}b^{23}} = a^2b^3\sqrt[6]{a^4b^6}$.

Method of performing this reduction.

In general to reduce two radical Quantities $\sqrt[m]{a^pb^q}$ and $\sqrt[n]{a^rb^s}$ to the same Exponent, the first must be transformed into $\sqrt[mn]{a^{pn}b^{qn}}$, and the second into $\sqrt[mn]{a^{rm}b^{sm}}$. If they are to be multiplied, their Product would be $\sqrt[mn]{a^{pn+rm}b^{qn+sm}}$, and if the first is to be divided by the second, the Quotient will be $\sqrt[mn]{a^{pn-rm}b^{qn-sm}}$.

It is easy to perceive that when the Exponents of the radical Quantities have a common Divisor, there is no Necessity of reducing each Radical into another, whose Exponent is the Product of the two first Exponents; for Example, if $\sqrt{ab}$ and $\sqrt[4]{ab^3}$ be proposed, the first will be transformed into $\sqrt[4]{a^2b^2}$; in like Manner if $\sqrt[4]{a^3b}$, $\sqrt[6]{a^6b}$ are transformed into $\sqrt[12]{a^9b^3}$ and $\sqrt[12]{a^{12}b^2}$.

XLI.

The foregoing Operations may be performed after another Manner, by conſidering the radical Quantities as Powers with fractional Exponents.

Thus to multiply $\sqrt[m]{a^p b^q}$ by $\sqrt[n]{a^r b^s}$ is the ſame Thing as to multiply $a^{\frac{p}{m}} b^{\frac{q}{m}}$ by $a^{\frac{r}{n}} b^{\frac{s}{n}}$, whoſe Product is $a^{\frac{p}{m}+\frac{r}{n}} b^{\frac{s}{n}+\frac{q}{m}}$, or $a^{\frac{pn+rm}{mn}} b^{\frac{sm+qn}{mn}}$. Another Method of performing the foregoing operations.

If it was propoſed to divide the firſt Quantity $\sqrt[m]{a^p b^q}$ by the ſecond $\sqrt[n]{a^r b^s}$ the Quotient would be $a^{\frac{p}{m}-\frac{r}{n}} b^{\frac{s}{n}-\frac{q}{m}}$ or $a^{\frac{pn-rm}{mn}} b^{\frac{sm-qn}{mn}}$.

If it was propoſed to divide $\sqrt[2p]{a^m b^{3n} c^2}$ by $\sqrt[p]{a^{2m} b^n c}$, transforming the firſt Quantity into $a^{\frac{m}{2p}} b^{\frac{3n}{2p}} c^{\frac{1}{p}}$, and the ſecond into $a^{\frac{2m}{p}} b^{\frac{n}{p}} c^{\frac{1}{p}}$, and ſubtracting the Exponents $\frac{2m}{p}$, $\frac{n}{p}$, $\frac{1}{p}$, of the Letters $a$, $b$, $c$ in the ſecond Quantity from the Exponents $\frac{m}{2p}$, $\frac{3n}{2p}$, $\frac{1}{p}$ of the ſame Letters in the firſt Quantity, the Quotient will be found to be

$$a^{-\frac{3m}{2p}} b^{\frac{n}{2p}} \times 1 = \frac{b^{\frac{n}{2p}}}{a^{\frac{3m}{2p}}}$$

XLII.

Compound Surds are ſuch as conſiſt of two or more joined together; ſimple Surds by being multiplied into themſelves give at length rational Quantities, yet compound Surds multiplied into themſelves, commonly give ſtill irational Products, but when any compound Surd is propoſed, there is another compound Surd, which multiplied into it, gives a rational Product; thus $\sqrt{a} + \sqrt{b}$ multiplied by $\sqrt{a} - \sqrt{b}$ gives $a - b$. whence Fractions affected with ſurd Quantities, may be reduced to a more ſimple Expreſſion, by multiplying the Denominator of the Fraction by that Surd which will give a rational Product, and multiplying the Numerator by the ſame Surd.

XLIII.

Let $a^m + bl$ expreſs any binomial Surd, and let it be propoſed to transform it into another Binomial, in which the Quantity $a$ ſhall have a given Exponent $n$, and the Quantity $b$ ſhall not be multiplied by $a$, that is,

let it be propoſed to find a multinomial, by which the propoſed Binomial being multiplied, the firſt Term of the Product, ſhall be $a^n$, and all the other Terms except the laſt, ſhall deſtroy each other.

Let $a^x\ b^y + a^z\ b^u$ be any two Terms of this Multinomial immediately ſucceeding each other, and let them be multiplied by the propoſed Binomial, the Product thence ariſing will be $a^{m+x}\ b^y + a^x\ b^{l+y} + a^{x+z}\ b^u + a^z\ b^{l+u}$, and the firſt Term of the Product ſhould be $a^n$, $m + x = n$, $x = n - m$ and $y = o$, wherefore the firſt Term of the Multiplicator is $a^{n-m}$.

Inveſtigation of the Surd which multiplied into a propoſed Surd gives a rational Product.

To find the ſecond Term, I ſubſtitute this firſt Term in the aſſumed Product, whence there reſults $a^n + a^{n-m}\ b^l + a^{m+z}\ b^u + a^z\ b^{l+u}$ but $+ a^{n-m}\ b^l + a^{m+z}\ b^u = o$. Conſequently the ſecond Term of the Multiplicator ſhould be negative, and becauſe $n - m = m + z$ & $l = u$, Conſequently $z = n - 2m$ and $2l = u + l$, this ſecond Term will be $- a^{n-2m}\ b^l$.

To find the third Term of the Multiplicator, I ſubſtitute this ſecond Term in the aſſumed Product, and there reſults $a^n - a^{n-m}\ b^l + a^{n-m}\ b^l + a^{n-2m}\ b^{2l}$. Let now the laſt Term of this Product, be put equal to the firſt Term of the aſſumed Product, becauſe thoſe Terms ſhould deſtroy each other. $+ a^{n-2m}\ b^{2l} + a^{m+x}\ b^y = o$, whence $n - 2m = m + x$, $x = n - 3m$ and $y = 2l$ and $l + y = 3l$, Conſequently the third Term of the Multiplicator will be $+ a^{n-3m}\ b^{2l}$. To find the fourth Term, I ſubſtitute the foregoing Values in the aſſumed Product, and there reſults $a^n - a^{n-m}\ b^l + a^{n-m}\ b^l - a^{n-2m}\ b^{2l} + a^{n-2m}\ b^{2l} + a^{n-3m}\ b^{3l} + a^{m+z}\ b^u + a^z\ b^{l+u}$ whence $n - 3m = m + z$ & $z = n - 4m$ and $u = 3l$ and $l + u = 4l$, Conſequently the fourth Term of the Multiplicator will be $- a^{n-4m}\ b^{3l}$, and in General, the $p$ Term will be $a^{n-pm}\ b^{(p-1)l}$.

Now let the $p$ Term be the laſt Term of the Multiplicator, ſince the laſt Term of the Product reſults from the Multiplication of the laſt Term of the Multiplicator into the laſt Term of the propoſed Binomial, and that it contains none of the Powers of $a$, it will be expreſſed by,

$b^{(p-1)l+l} = b^{\frac{nl}{m}}$ as $n - pm = o$ and of Courſe, $p = \frac{n}{m}$ and its Sign is poſitive only, when $\frac{n}{m}$ is an odd Number, ſince all the Terms of the Multiplicator expreſſed by even Numbers, are Negative.

XLIV.

Application of the foregoing Theorem for reducing Fractions involving ſurd Quan-

Whence in General, when any Quantity is divided by a Binomial Surd $a^m \pm b^l$ where $l$ and $m$ repreſent any Fractions whatſoever, by taking $n$, the leaſt integer Number, ſuch that $\frac{nl}{m}$ will be an Integer, and multiplying both Numerator and Denominator by $a^{n-m} \mp a^{n-2m}\ b^l + a^{n-3m}\ b^{2l} \mp a^{n-4m}\ b^{3l}$, &c. the Denominator of the Product will become rational

and equal to $a^{\frac{nl}{m}} \pm b^{\frac{nl}{m}}$ then dividing all the Members of the Numerator by this rational Quantity, the Quote arising, will be that of the propoſed Quantity divided by the Binomial Surd expreſſed in it's laſt Terms. tities to a more ſimple Expreſſion.

Thus $\frac{3}{\sqrt{5}-\sqrt{2}} = \frac{3\sqrt{5}+3\sqrt{2}}{3} = \sqrt{5}+\sqrt{2}$; $\frac{\sqrt{6}}{\sqrt{7}-\sqrt{3}} = \frac{\sqrt{42}+\sqrt{18}}{4}$ &c.

$$\frac{\sqrt[3]{20}}{\sqrt[3]{4}-\sqrt[3]{2}} = \frac{\sqrt[3]{20}}{\sqrt[3]{4}-\sqrt[3]{2}} \times \frac{\sqrt[3]{16}+2+\sqrt[3]{4}}{\sqrt[3]{16}+2+\sqrt[3]{4}} = \frac{\sqrt[3]{20}}{\sqrt[3]{4}-\sqrt[3]{2}} \times \frac{2\sqrt[3]{2}+2+\sqrt[3]{4}}{2\sqrt[3]{2}+2+\sqrt[3]{4}}$$

$$= \frac{2\sqrt[3]{40}+2\sqrt[3]{20}+\sqrt[3]{80}}{2} = 2\sqrt[3]{5} \times \sqrt[3]{20} \times \sqrt[3]{10}, \text{ alſo } \frac{\sqrt[2]{10}}{\sqrt[2]{2}-\sqrt[3]{3}}$$

$$= \frac{4\sqrt{20}+4\sqrt{10}\times\sqrt[3]{3}+2\sqrt{20}\times\sqrt[3]{9}+6\sqrt{10}+\sqrt[3]{20}\times\sqrt[3]{3}+3\sqrt{10}\times\sqrt[3]{9}}{-1}$$

$$= -8\sqrt{5}-4\sqrt{10}\times\sqrt[3]{3}-8\sqrt{5}\times\sqrt[3]{9}-6\sqrt{10}-6\times\sqrt{5}\times 3\sqrt{3}-3\sqrt{10}\times\sqrt{9}.$$

XLV.

To compleat the Reſolution of Equations, conſiſting of two Terms, it remains to explain how the unknown Quantity is diſengaged when it enters the Exponent.

Thus if it was required to find in what Time $x$ a Sum $a$ lent out at Intereſt at a given Rate *per Cent.* would amount Principle and Intereſt to a Sum $r$, the Equation to be ſolved would be $r = a\,p^x$ or $\frac{r}{a} = p^x$, in which Equation the unknown Quantity enters the Exponent.

XLVI.

Since the Exponent $x$ includes all determined Numbers if in the Room of $x$ be ſubſtituted ſucceſſively all the integer Numbers there will reſult. Inveſtigation of the Method of ſolving Equations conſiſting of two Terms when the unknown Quantity enters the Expone-

$\ldots p^{-7}, p^{-6}, p^{-5}, p^{-4}, p^{-3}, p^{-2}, p^{-1}, p^0, p^1, p^2, p^3, p^4$, &c. which forms a geometrical Progreſſion. Now, if between the Terms of this Progreſſion, there be inſerted a Number of Terms $n$, it is eaſy to perceive that they will be expreſſed by the Powers of $p$ with fractional Exponents, and that there will reſult a new Progreſſion, the Multiplicator being $q^{\frac{1}{n+1}}$ and that $n+1$ may be conceived ſo great and conſequently the Progreſſion may be conceived to increaſe ſo ſlowly, that all the Numbers from 1 to $\infty$, and from 1 to 0, will be found there, $p$ being ſuppoſed to be an Integer, and greater than Unity.

XLVII.

Let for Example $p = 10$, then the Progreſſion will be tranſformed into

$\ldots \frac{1}{10000000}, \frac{1}{1000000}, \frac{1}{100000}, \frac{1}{10000}, \frac{1}{1000}, \frac{1}{100}, \frac{1}{10}, 1, 10, 100, 1000, 10000, 100000$, &c.

9
89
009
000
189

Now, between 1 & 10, there are eight whole Numbers, 2, 3, 4, 5, 6, 7, 8, 9, and between 10 and 100, there are 89 whole Numbers, viz. 11, 12 &c. but if between each of the Terms of this Progreſſion, there be inſerted a Number of Terms $n$, there will reſult a new Progreſſion, the common Multiplicator being $10^{\frac{1}{n+1}}$ among whoſe Terms thoſe intermediate Numbers will be found.

To find the Term of this Progreſſion for Example, which is equal to 3, I proceed thus, the Number 3 required being one of the Terms of a geometrical Progreſſion, of which 1 & 10 are the Extreames; I inveſtigate the middle Term of this Progreſſion; now, this Term being equally diſtant from the Extreams 1 and 10, thoſe three Terms the firſt 1 the middle Term and the laſt Term 10 will be in continued Proportion, ſo that the middle Term will be equal to $\sqrt{1 \times 10}$ or $10^{,5}$, which I find to be 3, 16227766. The Number 3, 16227766 being greater than 3, I inveſtigate a new Term that will be the Mean of a continued geometrical Proportion, having for Extreams 1 & 3,16227766, which will be $\sqrt{1 \times 3,16227766}$ or $10^{,25} = 1,77827941$. As the Number 3 is greater than 1, 77827941, or $10^{,25}$ and leſs than 3,16227766 or $10^{,5}$, I inveſtigate a new Term, a mean proportional beween 1,77827941, and 3,16227766, which will be the ſquare Root of $10^{,25} \times 10^{,5}$ that is of 3,16227766 × 1, 77827941, or of 5,623413251 4809806, which is $10^{,375}$ or 2, 37137370. As the Number 3 is greater than the new Term, 2,37137370 and leſs than 3,16227766, I inveſtigate a mean Proportional between thoſe two Terms by extracting the ſquare Root of $10^{,375} \times 10^{,5}$ that is of 2,37137370 × 3,16227766 which is 2,73841962. As the Number 3 is greater than the new Term 2,73841962 and leſs than 3,16227766, I find a mean Proportional between thoſe two Terms by extracting the ſquare Root of $10^{,4375} \times 10^{,5}$ that is of 2,73841962 × 3,16227766 or of 8,659643188031 6892 which will be $10^{,46875}$ or 2,94272717. The Number 3 being ſtill greater than the laſt Term 2,94272717 and leſs than the firſt 3, 16227766, I inveſtigate a mean Proportional between theſe two Terms, by extracting the ſquare Root of $10^{,46875} \times 10^{,5}$, or of 9,305720389166 0222, which will be $10^{,484375}$, or 3,05052789.

Application of the foregoing Theorem for reducing Fractions involving ſurd Quan-

$\frac{a^m}{\pm a^n \pm}$

leaſt

both

Proceeding in this Manner to inveſtigate new Mean proportionals between two Terms, one greater and the other leſs than 3, after having performed 26 of thoſe Operations, I find $10^{,47712125} = 3,00000000$, which does not differ from 3 by a decimal Unit of the eighth Order, theſe 26 mean Proportionals are ſet down in the following Table, each of them being placed between the Extreams, and to indicate the Order of the Operations, the two firſt Extreams are marked by the Letters *A* and *B*, and the mean Term placed between them by *C*, the other Means are denoted by the following Letters of the Alphabet.

XLVII.

| | | | |
|---|---|---|---|
| A | $10^{0,00000000}$ | = | 1,00000000 |
| C | $10^{0,50000000}$ | = | 3,16227766 |
| B | $10^{1,00000000}$ | = | 10,00000000 |
| A | $10^{0,00000000}$ | = | 1,00000000 |
| D | $10^{0,25000000}$ | = | 1,77827941 |
| C | $10^{0,50000000}$ | = | 3,16227766 |
| D | $10^{0,25000000}$ | = | 1,77827941 |
| E | $10^{0,37500000}$ | = | 2,37137370 |
| C | $10^{0,50000000}$ | = | 3,16227766 |
| E | $10^{0,37500000}$ | = | 2,37137370 |
| F | $10^{0,43750000}$ | = | 2,73841962 |
| C | $10^{0,50000000}$ | = | 3,16227766 |
| E | $10^{0,34750000}$ | = | 2,73841962 |
| G | $10^{0,46875000}$ | = | 2,94272717 |
| C | $10^{0,50000000}$ | = | 3,16227766 |
| G | $10^{0,46875000}$ | = | 2,94272717 |
| H | $10^{0,48437500}$ | = | 3,05052789 |
| C | $10^{0,50000000}$ | = | 3,16227766 |
| H | $10^{0,48437500}$ | = | 3,05052789 |
| I | $10^{0,47656250}$ | = | 2,99614286 |
| G | $10^{0,46875000}$ | = | 2,94272717 |
| I | $10^{0,47656250}$ | = | 2,99614286 |
| K | $10^{0,48046875}$ | = | 3,02321308 |
| H | $10^{0,48437500}$ | = | 3,05052789 |
| K | $10^{0,48046875}$ | = | 3,02321308 |
| L | $10^{0,47851563}$ | = | 3,00964753 |
| I | $10^{0,47656250}$ | = | 2,99614286 |
| L | $10^{0,47851563}$ | = | 3,00964753 |
| M | $10^{0,47753906}$ | = | 3,00288762 |
| I | $10^{0,47656250}$ | = | 2,99612286 |
| M | $10^{0,47753906}$ | = | 3,00288762 |
| N | $10^{0,47705078}$ | = | 2,99951334 |
| I | $10^{0,47656250}$ | = | 2,99614286 |
| N | $10^{0,47705078}$ | = | 2,99951334 |
| O | $10^{0,47729492}$ | = | 3,00120000 |
| M | $10^{0,47753906}$ | = | 3,00288762 |
| O | $10^{0,47729492}$ | = | 3,00120000 |
| P | $10^{0,47717285}$ | = | 3,00035655 |
| N | $10^{0,47705078}$ | = | 2,99951334 |
| P | $10^{0,47717285}$ | = | 3,00035655 |
| Q | $10^{0,47711182}$ | = | 2,99993491 |
| N | $10^{0,47705078}$ | = | 2,99951334 |
| Q | $10^{0,47711182}$ | = | 2,99993491 |
| R | $10^{0,47714233}$ | = | 3,00014572 |
| P | $10^{0,47717285}$ | = | 3,00035655 |
| R | $10^{0,47714233}$ | = | 3,00014572 |
| S | $10^{0,47712708}$ | = | 3,00004031 |
| Q | $10^{0,47611182}$ | = | 2,99993491 |
| S | $10^{0,47712708}$ | = | 3,00004031 |
| T | $10^{0,47711945}$ | = | 2,99998751 |
| Q | $10^{0,47711182}$ | = | 2,99993491 |
| T | $10^{0,47711945}$ | = | 2,99998751 |
| V | $10^{0,47712326}$ | = | 3,00001390 |
| S | $10^{0,47712708}$ | = | 3,00004031 |
| V | $10^{0,47712326}$ | = | 3,00001390 |
| X | $10^{0,47712135}$ | = | 3,00000070 |
| T | $10^{0,47711945}$ | = | 2,99998751 |
| X | $10^{0,47712135}$ | = | 3,00000070 |
| Y | $10^{0,47712040}$ | = | 2,99999410 |
| T | $10^{0,47711945}$ | = | 2,99998751 |
| Y | $10^{0,47712040}$ | = | 2,99999410 |
| Z | $10^{0,47712088}$ | = | 2,99997739 |
| X | $10^{0,47712135}$ | = | 3,00000070 |
| Z | $10^{0,47712088}$ | = | 2,99999739 |
| A A | $10^{0,47712112}$ | = | 2,99999908 |
| X | $10^{0,47712135}$ | = | 3,00000070 |
| A A | $10^{0,47712112}$ | = | 2,99999908 |
| B B | $10^{0,47712123}$ | = | 2,99999989 |
| X | $10^{0,47712135}$ | = | 3,00000070 |
| B B | $10^{0,47712123}$ | = | 2,99999989 |
| C C | $10^{0,47712129}$ | = | 3,00000029 |
| X | $10^{0,47712135}$ | = | 3,00000070 |
| C C | $10^{0,47712129}$ | = | 3,00000029 |
| D D | $10^{0,47712126}$ | = | 3,00000009 |
| B B | $10^{0,47712123}$ | = | 2,99999989 |
| D D | $10^{0,47712126}$ | = | 3,00000009 |
| E E | $10^{0,47712125}$ | = | 3,00000000 |
| B B | $10^{0,47712123}$ | = | 2,99999989 |

By the ſame Method of Proceeding, the Terms correſponding to the other intermediate Numbers will be found, by inveſtigating mean Proportionals between 1 and 10, for the Numbers leſs than 10; between 10 & 100, for the Numbers leſs than 100, but greater than 10; between 100 & 1000, for the Numbers leſs than 1000 and greater than 100, &c.

XLVIII.

What is meant by the Logarithm of a Number.

Whence if $p$ be given, and if $p^x = b$, we can find the Value of $x$ ſuch that $p^x = b$: this Value of $x$ is called the Logarithm of $b$, and $p$ the Baſe of the Logarithm.

How the Logarithms of Numbers of one Syſtem are deduced from thoſe of another.

There are therefore as many different Syſtems of Logarithms, as there are different Numbers $p$ which may be aſſumed for the Baſe, but in two Syſtems the Logarithms of the ſame Number are always in a conſtant Ratio. Let the Baſe of one Syſtem $= p$, and of the other $= q$, and the Logarithm of the Number $b$ in the firſt Syſtem $= x$, & in the ſecond $= y$, then $p^x = b$ and $q^y = b$, wherefore $p^x = q^y$ conſequently $p = q^{\frac{x}{y}}$ whence the Fraction $\frac{x}{y}$ is always the ſame whatever Number is aſſumed for $b$. Whence if one Syſtem of Logarithms of all Numbers were computed, the Logarithms of any other Syſtem may be found by the Rule of Three. Thus if the Logarithms correſponding to the Baſe 10, were calculated, the Logarithms correſponding to any other Baſe for Example 3 may be found. Let the Logarithm $y$ of the Number $b$ correſponding to the Baſe 3 be required, the Logarithm $x$ of the ſame Number $b$ correſponding to the Baſe 10 being given. Since *Log.* 3, correſponding to the Baſe 10 $= 0{,}47712125$ and *Log.* 3, correſponding to Baſe 3 $= 1$. $0{,}47712125 : 1 = x : y$, wherefore $y = \frac{x}{0{,}47712125} = 2{,}095933 \times x$, if therefore all the Logarithms correſponding to the Baſe 10, be multiplied by 2,095933, there will reſult the Logarithms correſponding to the Baſe 3.

XLIX.

In every Syſtem the Logarithm of Unity is 0.

In every Syſtem of Logarithms, *Log.* $1 = 0$, for if in the Equation $p^x = b$, we put $b = 1$, then $x = 0$.

The Logarithms of Numbers greater than Unity, are poſitive, thus, *Log.* $p = 1$; *Log.* $p^2 = 2$; *Log.* $p^3 = 3$, &c. But the Logarithms of Numbers leſs than Unity, but poſitive, are negative, for *Log.* $\frac{1}{p} = -1$; *Log.* $\frac{1}{p^2} = -2$, *Log.* $\frac{1}{p^3} = -3$, &c. and the Logarithms of negative Numbers are imaginary.

The Logarithms of negative Numbers are imaginary.

In like Manner if *Log.* $p = x$, then *Log.* $p^2 = 2x$ *Log.* $p^3 = 3x$ and in general *Log.* $p^n = nx$ or *Log.* $p^n = n$ *Log.* $p$, on Account of $x =$ *Log.* $p$.

whence the Logarithm of any Power of $p$ is equal to the Logarithm of $p$ multiplied by the Exponent of the Power; thus $Log. \sqrt{p} = \frac{1}{2} x = \frac{1}{2} Log. p$, $Log. \frac{1}{\sqrt{p}} = Log. p^{-\frac{1}{2}} = -\frac{1}{2} Log. p$; and so on. But if $Log. a = y$. and $Log. b = x$: because $a = p^y$, and $b = p^x$, then $Log. ab = y + x$, whence the Logarithm of the Product of two Numbers, is equal to the Sum of the Logarithms of the Factors. In like Manner it will appear that $Log. \frac{a}{b} = y - x = Log. a - Log. b$, hence the Logarithm of a Fraction is equal to the Logarithm of the Numerator less the Logarithm of the Denominator.

The logarithm of the product of two numbers is equal to the Sum of the logarithms of the factors, and the logarithm of their quotient, to their difference.

Whence from the Equation $p^x = \frac{r}{a}$ is deduced $x = \frac{Log. r - Log. a}{Log. p}$ it being indifferent what System of Logarithms is employed (Art. XLVIII).

L.

The Resolution therefore of Exponential Equations, or of those in which the unknown Quantity enters the Exponent, is reduced to the Investigation of a general Formula for finding the Logarithm of any given Number.

Let $y$, $y'$, $y''$ express any three Terms of the Progression of Art. XLVI. immediately succeeding each other; we will have $y : y' = y' : y''$, or $y' : y = y'' : y'$, Consequently $y' - y : y = y'' - y' : y'$, or (putting $y' - y = dy$ and $y'' - y' = dy'$) $dy : y = dy' : y'$, wherefore $\frac{dy}{y} = \frac{dy'}{y'}$, whence $\frac{dy}{y}$ is a permanent Quantity.

Investigation of a general formula for finding the logarithm of any given number.

Let the Exponent of the Power of $p$ which is equal $y$ be expressed by $x$, the Exponent of the Power of $p$, which is equal $y'$ be expressed by $x + dx$, the Exponent of the Power of $p$ which is equal $y''$ will be expressed by $x + 2dx$, because the Exponents of the Terms of this Progression increase by the same constant Difference $dx$, whence assuming $M$ for a determined Quantity, $dx = \frac{M dy}{y}$, or $\frac{dx}{dy} = \frac{M}{y}$, consequently $\frac{M}{y}$ is the Value of the Ratio of the cotemporary Increments of any Number $y$ and of its Logarithm $x$.

To find the Ratio of the Quantities $y$ and $x$ from that of their Increments, I put $y = 1 + z$, and there results $dy = dz$, and $\frac{dx}{dy} = \frac{dx}{dz} = \frac{M}{1+z} = M \times (1+z)^{-1} = M(1 - z + z^2 - z^3 + z^4 - z^5 + \&c)$.

I multiply this Value of $\frac{dx}{dz}$ by $z$, and consequently there will arise $M(z - z^2 + z^3 - z^4 + z^5 - z^6 + \&c.)$ which Terms I divide severally by their Number of Dimensions, and the Result

$$M \times \left( z - \frac{1}{2} z^2 + \frac{1}{3} z^3 - \frac{1}{4} z^4 + \frac{1}{5} z^5 - \frac{1}{6} z^6 + \text{&c.} \right).$$

I put $= x$, to which Equation no conſtant Quantity is to be added, becauſe when $z = 0$ it is reduced to 0, as it really ſhould, for when $z = 0$ $y = 1 + z = 1$, whoſe Logarithm $= 0$. And by this Equation is expreſſed the Ratio between $x$ & $y$, ſince from this Equation we can return back to the propoſed Equation.

LI.

Inconveniency to which the foregoing method is liable.

By this Formula, the Logarithm of any Number greater than Unity, but leſs than 2 may be found; I ſay leſs than 2, for if $1 + z$ is equal 2, then the two Terms of the Denominator will be equal, and the Series will be erroneous, and much more ſo when $z$ exceeds 2, however, the foregoing Formula may ſerve for finding the Logarithm of any Number greater than Unity, by calculating the Logarithms of ſuch Numbers that are leſs than 2, which multiplied into one another, or divided by each other, produce the propoſed Number.

How it may be removed.

For Example, finding that $\frac{\frac{12}{10} \times \frac{12}{10}}{\frac{8}{10} \times \frac{9}{10}} = \frac{1,2 \times 1,2}{0,8 \times 0,9} = 2$,

I calculate by the foregoing Formula the Logarithm of 1, 2, putting $z = 0, 2$; the Logarithm of 0, 8 ſuppoſing $z = -0, 2$, and the Logarithm of 0, 9 by putting $z = -0, 1$. I add together the Logarithms of 0, 8 & 0, 9, and deduct their Sum from the Double of the Logarithm of 1, 2, and there reſults $M \times 0,693147180559$, &c. for the Logarithm of 2 required.

To determine the Value of $M$ correſponding, for Example, to the Syſtem of Logarithms whoſe Baſe is 10, I firſt inveſtigate the Logarithm of 10, finding that $\frac{2 \times 2 \times 2}{0,8} = 10$, I triple the Logarithm already found, and deduct from it the Logarithm of 0, 8, the Remainder will be $M \times 2,302585092994$, &c. is the Logarithm of 10; now the Logarithm of 10 correſponding to the Baſe 10 is 1, and of Courſe we will have $1 = M \times 2,302585092994$, wherefore the Value of $M$ correſponding to the Syſtem of Logarithms, whoſe Baſe is 10, will be 0, 43429448, &c.

LII.

Another method for finding the logarithm of any number

The foregoing Method of finding the Logarithms of Numbers being not very ready, we ſhall proceed to explain how the Analyſts have remedied this Inconveniency.

Let the Sum of any two Numbers be expreſſed by $z$, and their Difference by $v$, conſequently the greateſt will be $\frac{1}{2} z + \frac{1}{2} v$, and the leaſt $\frac{1}{2} z - \frac{1}{2} v$. Now let $y = \frac{\frac{1}{2} z + \frac{1}{2} v}{\frac{1}{2} z - \frac{1}{2} v} = \frac{z + v}{z - v}$, and ſuppoſing $x$

to be a permanent Quantity, we will have $dy = \frac{2z \cdot dv}{(z-v)(z-v-dv)}$; Wherefore the Equation. $\frac{dx}{dy} = \frac{M}{y}$ will be transformed into $\frac{dx \times (z-v)(z-v-dv)}{2z \cdot dv} = \frac{M \times (z-v)}{z+v}$ or into $\frac{dx}{dv} = \frac{2z \cdot M}{(z+v) \times (z-v-dv)}$, Wherefore $\frac{2z \cdot M}{(z+v) \times (z-v-dv)}$ expreſſes the Ratio of any Cotemporary Increments of $x$ and $v$, and this Ratio is always greater than $\frac{2z \cdot M}{(z+v)(z-v)}$, but $dv$ decreaſing, this Quantity will alſo decreaſe, and as we may take $dv$ as ſmall as we pleaſe, we may make $\frac{2z \cdot M}{(z+v) \times (z-v-dv)}$ approach as near as we pleaſe to $\frac{2z \cdot M}{(z+v)(z-v)}$ wherefore $\frac{2z \cdot M}{(z+v)(z-v)}$ is the Limit of the Ratio $\frac{dx}{dv}$.

To find the Ratio of the Quantities $x$ and $v$, from the limiting Ratio of their cotemporary Increments $\frac{2z \cdot M}{(z+v)(z-v)} = \frac{2Mz}{z^2-v^2} =$ $2 \cdot M \cdot z \cdot (z^2-v^2)^{-1} = 2M\left(\frac{1}{z} + \frac{v^2}{z^3} + \frac{v^4}{z^5} + \frac{v^6}{z^7} + \&c\right)$, I multiply this Value of $\frac{dx}{dv}$ by $v$, and conſequently there will ariſe $2M\left(\frac{v}{z} + \frac{v^3}{z^3} + \frac{v^5}{z^5} + \frac{v^7}{z^7} + \&c\right)$, which Terms I divide ſeverally by the Number of their Dimenſions, and I put the Reſult $2M\left(\frac{v}{z} + \frac{v^3}{3z^3} + \frac{v^5}{5z^5} + \frac{v^7}{7z^7} + \&c\right)$ equal $x$, to which Equation no conſtant Quantity is to be added, becauſe when $v = 0$ it is reduced to 0, as it really ſhould, for when $v = 0$, the Fraction $\frac{z+v}{z-v}$ is reduced to $\frac{z}{z} = 1$, whoſe Logarithm $= 0$.

LIII.

Whence if the Sum of two Numbers be expreſſed by $z$, and their Difference by $v$, and if $\frac{2Mv}{z}$ be put $= A$, $\frac{Av^2}{z^2} = B$, $\frac{Dv^2}{z^2} = C$, $\frac{Cv^2}{z^2} = D$, &c. the Logarithm of the Quotient of the greateſt divided by the leaſt will be expreſſed by $A + \frac{1}{3}B + \frac{1}{5}C + \frac{1}{7}D + \&c$.

For Example, putting $M = 1$, the Logarithm $p$ of the Quotient of 126 divided by 125 may be found, the Logarithm $q$ of the Quotient of 225 divided by 224, the Logarithm $r$ of the Quotient of 2401 divided by 2400, and the Logarithm $\int$ of the Quotient of 4375 divided by 4374.

But $p = Log.\ 126 - Log.\ 125 = Log.\ 2.\ 3^2.\ 7 - Log.\ 5^3 = Log.\ 2 + 2\ Log.\ 3 - 3\ Log.\ 5 + Log.\ 7$

$q = Log.\ 225 - Log.\ 224 = Log.\ 3^2.\ 5^2 - Log.\ 2^5.\ 7 = -5\ Log.\ 2 + 2\ Log.\ 3 + 2\ Log.\ 5 - Log.\ 7.$

$r = Log.\ 2401 - Log.\ 2400 = Log.\ 7^4 - Log.\ 2^5.\ 3.\ 5^2 = -5\ Log.\ 2 - Log.\ 3 - 2\ Log.\ 5 + 4\ Log.\ 7.$

$\int = Log.\ 4375 - Log.\ 4374 = Log.\ 5^4.\ 7 - Log.\ 2.\ 3^7 = -Log.\ 2. - 7\ Log.\ 3 + 4\ Log.\ 5 + Log.\ 7.$

Whence $a\,p = a\ Log.\ 2 + 2a\ Log.\ 3 - 3a\ Log.\ 5 + a\ Log.\ 7.$

$b\,q = -5b\ Log.\ 2 + 2b\ Log.\ 3. + 2b\ Log.\ 5. - b\ Log.\ 7.$

$c\,r = -5c\ Log.\ 2 - c\ Log.\ 3 - 2c\ Log.\ 5 + 4c\ Log.\ 7.$

$d\int = -d\ Log.\ 2 - 7d\ Log.\ 3 + 4d\ Log.\ 5 + d\ Log.\ 7.$

Let $a, b, c, d$ be such, that $ap + bq + cr + d\int = Log.\ 2 + Log.\ 5$, whence arises the following Equations for determining those Quantities, $a - 5b - 5c - d = 1$; $2a + 2b - c - 7d = 0$; $-3a + 2b - 2c + 4d = 1$; $a - b + 4c + d = 0$; which give $a = 239$, $b = 90$, $c = -63$, $d = 103$; wherefore $239\,p + 90\,q - 63\,r + 103\int$, or 2, 302585092924, $= Log.\ 10$, the same as before, and the Value of $M$ corresponding to the System whose Base is 10, will be found to be 0,434294481903, and substituting for $M$ this Value, there will result

$$M(202p + 76q - 53r + 87\int) = Log.\ 7.$$
$$M(167p + 63q - 44r + 72\int) = Log.\ 5.$$
$$M(114p + 43q - 30r + 49\int) = Log.\ 3.$$
$$1 - M(167p + 63q - 44r + 72\int) = Log.\ 2.$$

Having $M$, and the Logarithms of the prime Numbers 2, 3, 5, 7 below 10, to obtain the Logarithms of the prime Numbers 11, 13, 17, 19 &c. Above 10, let $A$ be any prime Number, I suppose that the Logarithm of all the prime Numbers that precede $A$ have been found, then

$$Log.\ A = \frac{1}{2}\left[ Log.\ \frac{A^2}{A^2 - 1} + Log.\ (A - 1) + Log.\ (A + 1). \right].$$

If the Logarithms of all the prime Numbers that precede $A$, are not known, let the Logarithm of any other Number $B$ be given. Find the Logarithm of the Fraction $\frac{A}{B}$ if $A$ be greater than $B$, or of $\frac{B}{A}$ if it is less: This Logarithm being found, add it, in the first Case, to the given Lo-

garithm of $B$, and there will result the Logarithm of $\frac{A}{B} \times B$, that is, of $A$.

In the second Case, deduct the Logarithm of the Fraction $\frac{B}{A}$ from the given Logarithm of $B$, and there will result the Logarithm of $\frac{B}{\frac{B}{A}} = B \times \frac{A}{B} = A$.

LIV.

Application of the foregoing formulas to an example.

To shew the Use of these Formulas by an Example, let it be required to find in what Time 575 *l.* will raise a Stock of 756 *l.* 13 *s.* 2 *d.* $\frac{1}{4}$ at 4 *per Cent.*

In this Case we have $p = 1,04$, $a = 575$, and $r = 756,66$. Whence $x = \frac{Log.\ 756,66 - Log.\ 575}{Log.\ 1,04} = 7$ the Number of Years required.

LV.

The Logarithms of numbers corresponding to the base 10 the best adapted to numeral arithmetick and why.

The Logarithms corresponding to the Base 10, besides their Utility which they have in common with other Systems, are the best adapted to numeral Arithmetick. For since the Logarithms of all Numbers except of the Powers of 10 are expressed in decimal Fractions, the Logarithms of Numbers between 1 and 10, will be contained within the Limits 0 and 1, the Logarithms of the Numbers between 10 & 100 will be contained within the Limits 1 and 2, and so on. Every Logarithm therefore is made up of an Integer and a decimal Fraction, the Integer is called the Characteristick, and the decimal Fraction the Mantissa. The Characteristick consists of as many Units less one as there are Places of Figures in the Number to which it corresponds, the Characteristick, for Example, of the Logarithm of the Number 78509 will be 4, because it consists of five Figures. Whence, from the Logarithm of any Number it is perceivable, how many Places of Figures it consists of: Thus the Number corresponding to the Logarithm 7,5804631 will consist of 8 Places of Figures.

LVI.

If the Logarithms differ only by their Characteristicks, the Numbers to which they correspond, will be to each other as a Power of 10 to 1, and consequently will be expressed by the same significative Figures, thus the Numbers corresponding to the Logarithms 4,9130187, and 6,9130187 will be 81850 & 8185000, the Number corresponding to the Logarithm 3,9130187 will be 8185, and the Number corresponding to the Logarithm 0,9130187 will be 8,185: The Mantissa therefore indicates the Figures by which the Number is expressed, and the Characteristick shews how many Figures to the right Hand are to be separated by a Comma. Thus, if the Lo-

garithm 2, 7603429 was found, the Mantiſſa indicates thoſe Figures 5758945, and the Characteriſtick 2 determines the Number correſponding to this Logarithm to be 575,8945. if the Characteriſtick was 0, the Number correſponding would be 5, 758945, if the Characteriſtick was — 1, the Number correſponding would be ten Times leſs, viz. 0,57589450, and if the Characteriſtick was — 2, the Number correſponding would be ten Times leſs, viz. 0,05758945, &c. in the Room of thoſe negative Characteriſticks, — 1, — 2, — 3, &c. the Analyſts ſet down 9, 8, 7, &c. it being underſtood that thoſe Logarithms are to be diminiſhed by 10.

LVII.

Uſe of logarithms for performing numeral operations.

The Tables of Logarithms are of great Uſe for performing with eaſe, and Expedition the Operations in numeral Arithmetick, becauſe, by the help of thoſe Tables, not only the Logarithm of any given Number, but alſo the Number correſponding to any given Logarithm may be found.

First example.

*A Debt for Example of a* 1000 *l.* 3 *Years due, was acquitted Principal and Intereſt, by paying* 1191 *l.* $\frac{2}{125}$ *let the Rate of Intereſt allowed be required.*

Here $a = 1000\,l.$ $r = 1191\,l.\ \frac{2}{125}$, $x = 3$, are given to find $p$. Conſequently we will have $p = \sqrt[x]{\frac{r}{a}}$, or Log. $p = \frac{Log.\ r - Log.\ a}{x}$ $= \frac{3,0759179 - 3,0000000}{3} = \frac{0,0759179}{3} = 0,0253059$ : Since 0, 0253059 is the Logarithm of $p$ or of $\frac{d+i}{d}$, adding to it the *Log.* of $d$ or of 100, the Sum 2, 0253059 is the *Log.* of $d + i$, but to this Logarithm correſponds the Number 106, wherefore $d + i = 106$, wherefore $i = 6$, the rate of Intereſt required.

Second example.

*The Sum of* 1000 *l. was borrowed at* $7\frac{1}{2}$ *per Cent. per Annum, the Sum due at the End of three Years ſeven Months and* 15 *Days are required.*

Here $a = 1000$, $i = 7\frac{1}{2}$, $d = 100$, $p = \frac{107\frac{1}{2}}{100} = \frac{215}{200} = \frac{43}{40}$; $x = \frac{1320}{365} = \frac{264}{73}$ Years, conſequently $r = p^x$, or *Log.* $r =$ *Log.* $a + x$ *Log.* $p. = 3,0000000 + \frac{264}{73} \times 0,0314085$ $= 3,0000000 + 0,1135869 = 3,1135869$ to which Logarithm correſponds the Number $1298\frac{29}{30}$ which expreſſes the Sum due required.

LVIII.

If it be agreed between the Borrower and the Lender, that the Intereſt tho' not paid when it becomes due, is not to be added to the Principle, ſo that the Amount ſhall be converted into a new Principle for the ſucceeding Term; the Sum $r$ due at the Expiration of any limited Time will be expreſſed by $a + \frac{a\,t\,i}{d} = a\left(1 + \frac{t\,i}{d}\right)$ and in this Caſe the Lender is ſaid to receive ſimple Intereſt for the Loan of his Money.

In what caſes compound intereſt is favourable to the borrower, and in what caſes diſadvantageous.

Now comparing this Expreſſion $a\left(1 + \frac{t\,i}{d}\right)$ of the Sum due, at the Expiration of a Number of Years $t$, computed (in the Language of the Analyſts) at ſimple Intereſt, with the former Expreſſion $a\left(\frac{d+i}{d}\right)^t$ of the Sum due at the Expiration of the ſame Number of Years, computed at compound Intereſt, it will appear 1° that if $t$ is an Integer greater than Unity, $\left(\frac{d+i}{d}\right)^t > \left(1 + \frac{t\,i}{d}\right)$ for $\left(\frac{d+i}{d}\right)^t = \frac{d^t}{d^t} + \frac{t\,i\,d^{t-1}}{d^t} + \frac{t.t-1.i^2\,d^{t-2}}{2.d^t} + \frac{t.t-1.t-2.i^3\,d^{t-3}}{2\times 3\,d^t}$ + &c.

Now this Quantity is manifeſtly equal to $1 + \frac{t\,i}{d}$ + a real and poſitive Quantity, conſequently it is greater than $\left(1 + \frac{t\,i}{d}\right)$.

2° if $t = 1$ it will eaſily appear that thoſe two Quantities will be equal.

3° if $t = \frac{1}{p}$ we will have $\left(\frac{d+i}{d}\right)^{\frac{1}{p}} < 1 + \frac{i}{d}$ or $< 1 + \frac{i}{dp}$ for raiſing the Quantities on both Sides to the Power $p$ there will reſult on one Side $\frac{d+i}{d}$, and on the other $1 + \frac{i}{d}$ + a real and poſitive Quantity.

4° from whence we may conclude in general, that if $t$ is any fractional Number greater than Unity, $a\left(\frac{d+i}{d}\right)^t > a + \frac{t\,i\,a}{d}$, and if $t$ is a Fraction leſs than Unity, $a\left(\frac{d+i}{d}\right)^t > a + \frac{t\,i\,a}{d}$.

Whence it appears, that when compound Intereſt is allowed for the Loan of Money, the Sum due at the Expiration of any Time exceeding a Year, is greater than it would be if ſimple Intereſt was allowed, that the Sum due at the Expiration of any Time leſs, is leſs than it would be, if ſimple Intereſt was allowed, wherefore, if compound Intereſt is advantageous to the Lender,

in certain Caſes it is alſo to the Borrower in others, the Compenſation it is true is not equal, ſince the Advantage of the Borrower ends with the firſt Year, and that of the Lender goes on continually increaſing as the Number of Years increaſes.

LIX.

Problem. *A Sum* a *was lent, on Condition that the Principal and Intereſt at a certain Rate* i *per Cent per Annum, ſhould be diſcharged in a Number* n *of equal Payments, one at the end of every Year; and that at each Payment, the Intereſt then due ſhould be cleared, and the Remainder by which ſuch Payment exceeds the Intereſt applied to reduce the Principal, the Value of the Payments is required.*

To ſolve this Queſtion, I obſerve that the firſt Payment is compoſed of two Parts, one of which is the Intereſt of the Principal at the ſuppoſed Rate, due at the End of the firſt Year; the other is a Portion of the Principal taken to compleat the Payment. That the Principal being diminiſhed by the firſt Payment, the Intereſt due at the End of the ſecond Year will be leſs than the former, and conſequently the Portion of the Principal to be taken to compleat the ſecond Payment, will be greater than the Portion taken to compleat the firſt, and ſo on for the ſucceeding Years. Whence there reſults two Series, the one decreaſing, whoſe Terms expreſs the Intereſt due at the End of every Year, the other increaſing whoſe Terms repreſent the different Portions of the Principal taken to compleat the Payments. It is this laſt Series we ſhall conſider, and to diſcover the Law of it, let $z$, $y$, $x$ &c. expreſs the Portions of the Principal taken to compleat the firſt, ſecond, third &c. Payments, ſo that $z + y + x +$ &c. $= a$.

The firſt Payment will be $\frac{a\,i}{d} + z$, the 2$^d$ $\frac{a\,i - z\,i}{d} + y$, the Third $\frac{a\,i - z\,i - y\,i}{d} + x$ &c. As thoſe Payments are ſuppoſed to be equal, by comparing the firſt with the ſecond, the ſecond with the third, &c. there will reſult ſo many different Equations. Whence is deduced $y = z \times \frac{d+i}{d}$, $x = z \times \left(\frac{d+i}{d}\right)^2$, &c. from whence it appears, that the Series in Queſtion is a geometrical Progreſſion, which by putting $\frac{d+i}{d} = p$, will be expreſſed as follows, $z$, $z\,p$, $z\,p^2$, $z\,p^3$ &c. $z\,p^{n-3}$, $z\,p^{n-2}$, $z\,p^{n-1}$. To find the firſt Term of this Progreſſion, I obſerve that the Sum of all the Terms except the firſt, is equal to the Sum of all the Terms except the laſt, multiplied by the common Ratio, wherefore $a - z = a\,p - z\,p^n$ whence $a\,p - a = z\,p^n$ or $z = a + \frac{p-1}{p^n - 1}$, to which adding the Intereſt of

the Capital due at the End of the first Year, which is $a \times (p - 1)$, there results $r = a \times \left(\frac{p-1}{p^n-1} + p - 1\right)$, or (reducing the whole to the same Denominator) $r = a \times \left(\frac{p^{n+1} - p^n}{p^n - 1}\right)$, or in Logarithms, $Log.\ r = Log.\ (p^{n+1} - p^n) - Log.\ (p^n - 1)$.

Application of the foregoing problem to an example.

Let the Sum lent, for Example be 120 *l.* 5 *s.* Interest 6 *per Cent.* and to be reimbursed in 7 equal Payments. Here $a = 120,25$, $p = 1,06$, $n = 7$ whence $Log.\ r = Log.\ 10,84852 - Log.\ 0,50361 = 1,035369 - 9,702113 = 1,333256$, whence $r = 21\ l.\ 10\ s.\ 9\frac{1}{2}\ d.$

LX.

When a Debt is discharged by many equal Payments, and the Interest due at the Time of each Payment is cleared, before any Part of the Principal, compound Interest is allowed to the Lender. To make this appear, I argue thus, a Person who borrows a Sum $a$ will owe at the End of the first Year, $a \times \frac{d+i}{d}$, but by Hypothesis he reimburses at the End of this Year a Sum $r$, wherefore he will owe at the Beginning of the second Year $a \times \left[\frac{d+i}{d}\right] - r$, and at the End of the second Year he will owe $a \times \left[\frac{d+i}{d}\right] - r$ (considered as Principal) multiplied by $\frac{d+i}{d}$, that is, $a \times \left[\frac{d+i}{d}\right]^2 - r \times \frac{d+i}{d}$: and since at the End of the Year he reimburses a Sum $r$, he will owe at the Beginning of the the third Year $a \left[\frac{d+i}{d}\right]^2 - r \times \frac{d+i}{d} - r$, and at the beginning of the 4th Year $a \times \left[\frac{d+i}{d}\right]^3 - r \left[\frac{d+i}{d}\right]^2 - r \times \frac{d+i}{d} - r$, and in general at the End of the $n^{th}$ Year.

When a debt is discharged by several equal payments compound interest is allowed to the lender.

$$a \times \left[\frac{d+i}{d}\right]^n - r\left[\frac{d+i}{d}\right]^{n-1} - r\left[\frac{d+i}{d}\right]^{n-2} \ldots - r$$

Wherefore, if the Payment should be made in a Number $n$ Years, the foregoing Quantity must be put equal to nothing. Now I observe, that in this last Quantity all the Terms that are multiplied by $r$ form a geometrical Progression, of which the first Term is $\left[\frac{d+i}{d}\right]^{n-1}$, and the last Term 1. Now, the Sum of this Progression, putting for Brevity's sake, $\frac{d+i}{d} = p$ will be equal to $p^n - 1$ divided by $p - 1$, consequently $a p^n - r \times \frac{p^n - 1}{p - 1} = 0$, consequently $r = a \times \frac{p^{n+1} - p^n}{p^n - 1}$; the same Value as found before.

LXI.

What is meant by annuities.

When a Sum of Money is lent on Condition that the Borrower shall discharge both Principal and Interest in a Number of equal Payments to be made yearly, such Payments is called an Annuity. In Computations relating to Annuities therefore, there are four Things to be considered, the Sum lent $a$, the Rent received every Year $r$, the Rate *per Cent* $i$, the Number of Years $n$ that the Annuity is paid; and any three of those four Things being given, the other may be found, as will appear by the following Examples.

Let it be required to find what a yearly Rent $r$ is worth in ready Money for its Continuance $n$ Years, the Rate $i$ *per Cent per Annum*, compound Interest being allowed to the Purchaser.

Here $r$, $n$, $p$ are given to find $a$, which is equal to $r \times \frac{1 - \frac{1}{p^n}}{p - 1}$, or

Resolution of the various questions relating to annuities.

$$Log.\ \left(r - \frac{r}{p^n}\right) - Log.\ (p - 1).$$

To apply this general Solution to an Example, let the Rent be 30*l.* *per Annum*, to be continued 7 Years, and let the present Worth be required, allowing 6 *per Cent* compound Interest to the Purchaser.

Here we have $r = 30$, $n = 7$, and $p = 1{,}06$ and of Course $Log.\ a = Log.\ 10{,}048 - Log.\ 0{,}06 = 1{,}002079 - 8{,}778151 = 2{,}223928$ and consequently $a = 167\ l.\ 9\ s.\ 5\ d.$

LXII.

Let it be required to find for what Time $n$ a Sum $a$ will purchase an Annuity of $r$ Pounds *per Annum*, at the Rate $i$ *per Cent* compound Interest.

Here $a$, $r$, & $p$ are given to find $n$. Because $p^n = \frac{r}{r + a - ap}$

$$n = \frac{Log.\ r - Log.\ (r + a - ap)}{Log.\ p}$$

Let it be required to find, for Example, for what Time will 167*l.* 9*s.* 5*d.* purchase an Annuity of 30*l.* *per Annum*, at 6 *per Cent* compound Interest. In this Case $a = 167{,}416$, $r = 30$, $p = 1{,}06$, hence $n = \frac{Log.\ 30 - Log.\ 19.9517}{Log.\ 1{,}06}$ $= 7$ Years, the Time required.

LXIII.

Annuities are said to be in Arrears when they are payable or due either yearly or half yearly, &c. and are unpaid for any Number of Payments, to compute the Amount $m$ of all those Payments, allowing any Rate of Interest for their Forbearance.

Suppose $r$ the first Years Rent, then $r\,p + r$ will be the Amount of the first Years Rent more the second Years Rent, and $r\,p^2 + r\,p + r$ the Amount of the first and second Years Rent, more the third Years Rent, and in general the Amount of the $n - 1$ Years Rent more the $n$ Years Rent will be expressed by $r\,p^{n-1} + r\,p^{n-2} + r\,p^{n-3}$ . . . . $+ r$.

Now the Sum of this Progreſſion is $\frac{r p^n - r}{p-1} = m$ the Amount required, or $Log.\ m = Log.\ (r p^n - r) - Log.\ (p-1)$.

LXIV.

To apply this Formula to an Example, let a yearly Rent of 30 *l.* be unpaid 9 Years, and let its Amount at the Rate of 6 per Cent per Annum, compound Intereſt be required.

Reſolution of the various queſtions relating to annuities in arrears.

Here $r = 30$, $n = 9$ & $p = 1,06$ & $Log.\ m = Log.\ 20,68353 - Log.\ 0,06 = 1,315626 - 8,778151 = 2,537475$, and conſequently $m = 344 l.\ 14 s.\ 6\frac{1}{2} d.$

LXV.

*Let it be required to find what Annuity* r *forborn* n *Years, will raiſe a Stock* m, *at a Rate* i per Cent *compound Intereſt.*

Here $m$, $p$, $n$ are given to find $r$, which is equal to $\frac{m \times (p-1)}{p^n - 1}$, or $Log.\ r = Log.\ (m\,p - m) - Log.\ (p^n - 1)$.

Let $a = 344\ l.\ 5\ s.$ $n = 8$, and $i = 3\ l.\ 10\ s.$ Conſequently $p = 1,035$, wherefore $Log.\ r = Log.\ 12,04875 - Log.\ 0,316803 = 1,080908 - 9,500785 = 1,580123$, and conſequently $r = 38,0297\ l. = 38\ l.\ 0 s.\ 7 d.$ *per Annum.*

LXVI.

Let it be required to find in what Time $n$, an Annuity of $r$ pounds *per Annum*, will raiſe a Stock $m$, at a Rate $i$ *per Cent.* Compound Intereſt.

Here $m$, $p$, $r$ are given to find $n$, which is equal to $\frac{Log.\ (m\,p + r - m) - Log.\ r}{Log.\ p.}$

Let $m = 344,25$, $r = 38,0297$, and $i = 3\ l.\ 10\ s.$ hence $p = 1,035$ and $n = \frac{Log.\ 50,07845 - Log.\ 38,0297}{Log.\ 1,035} = 8$ Years, the Time required.

LXVII.

As all freehold or real Eſtates may be conſidered as Annuities to continue forever, to obtain their true Value $a$, it ſuffices to put the Quantity $a p^n - r p^{n-1} - r p^{n-2} - r p^{n-3} - r p^{n-4} \ldots - r$, or $a - \frac{r}{p} - \frac{r}{p^2} - \frac{r}{p^3} \ldots - \frac{r}{p^n}$ continued to an infinite Number of Terms equal to nothing, which on Account of all the Terms that are multiply'd by $r$ being a geometrical Progreſſion infinitely decreaſing, and conſequently equal to $\frac{1}{p-1}$ is reduced to $a - \frac{r}{p-1} = 0$ which gives $a = \frac{r}{p-1}$.

*Let it be required, for Example, to find the Value of a freehold Eſtate of* 25 l. per Annum, *allowing* 5 *l.* 10 *s.* per Cent, *&c. Compound Intereſt to the Buyer.*

Here are given $r = 25$ l. $i = 5$ l. 10s. and consequently $p = 1{,}055$ wherefore, $a = 454$ l. 10s. $10\frac{1}{2}$d. the Value required.

LXVIII.

After having solved all the Difficulties that could occur in the Solution of Problems of every Degree, producing Equations consisting of two Terms, Order requires that we should proceed to explain how to solve Problems producing Equations of every Degree consisting of three Terms, but the Analysts have not as yet found a general Method for every Problem of this Kind, the following in its full Extent, produces Equations of every Degree, consisting of three Terms that have been compleatly solved.

LXIX.

Problems producing equations consisting of three terms solved by the method for equations of the second degree.

*Let it be required to find two Numbers, whose Product is* a, *and the Difference of their* $n^{th}$ *Powers is* b.

To solve this Question let the lesser Number be $x$, then the greater is $\frac{a}{x}$, Now, by the Question $b = \frac{a^n}{x^n} - x^n$, or $b\,x^n = a^n - x^{2n}$, consequently $x^{2n} + b\,x^n = a^n$, to resolve this Equation, I add the Square of $\frac{b}{2}$ to both Sides, and there results $x^{2n} + b\,x^n + \frac{1}{4}b^2 = a^n + \frac{1}{4}b^2$ whose Root is $x^n + \frac{1}{2}b = +\sqrt{(a^n + \frac{1}{4}b^2)}$, & $x^n = -\frac{1}{2}b + \sqrt{(a^n + \frac{1}{4}b^2)}$, wherefore $x = \sqrt[n]{[-\frac{1}{2}b \pm \sqrt{(a^n + \frac{1}{4}b^2)}]}$.

By means of this Formula, all Equations may be resolved consisting of three Terms, in the first of which, the Index of the unknown Quantity $x$, is double of the Index of $x$ in the second Term, and the third Term is a given Quantity, and it is easy to perceive, that all Equations in this general Formula $x^{2n} + bx^n = a^n$ cannot have more than four real Roots, and will only have two when $n$ is an odd Number.

LXX.

Examlpe of the foregoing method.

To apply this general Solution to an Example, let it be required to find two Numbers whose Product is 15, and the Difference of their Squares is 16.

Here $a = 15$, $b = 16$, and $n = 2$, consequently
$x = \pm\sqrt{[-8 \pm \sqrt{289}]} = \sqrt{\pm\sqrt{-8 \pm 17}} = +\sqrt{9} = 3$.

LXXI.

Another example.

Let $x^4 - b\,b\,x^2 = b\,b\,c\,c$ be proposed to be solved, adding on both Sides $\frac{1}{4}b^4$ Square of one half of the Coeficient of $x\,x$, there will result $x^4 - b\,b\,x\,x + \frac{1}{4}b^4 = \frac{1}{4}b^4 + b\,b\,c\,c$, whose square Root is $x^2 - \frac{1}{2}b\,b = \pm b\sqrt{(\frac{1}{4}b\,b + c\,c)}$, whence is deduced $x\,x = \frac{1}{2}b\,b \pm b\sqrt{(\frac{1}{4}b\,b + c\,c)}$, and consequently $x = \pm\sqrt{[\frac{1}{2}b\,b \pm b\sqrt{(\frac{1}{4}b\,b + c\,c)}]}$ susceptible, of two real Values, and of two imaginary ones. The two first are $x = \pm\sqrt{[\frac{1}{2}b\,b + b\sqrt{(\frac{1}{4}b\,b + c\,c)}]}$, the two others are

$x = \pm\sqrt{[\frac{1}{2}bb - b\sqrt{(\frac{1}{4}bb + cc)}]}$ which are neceſſarily imaginary, becauſe $b\sqrt{(\frac{1}{4}bb + cc)}$ is greater than $\frac{1}{2}bb$.

LXXII.

Let the Equation $x^6 - 2a^2bx^3 = a^6$ be propoſed, adding to both Sides $a^4bb$, there reſults $x^6 - 2aabx^3 + a^4bb = a^4bb + a^6$, whoſe ſquare Root is $x^3 - aab = a^2\sqrt{(a^2 + b^2)}$, or $x^3 = a^2b \pm a^2\sqrt{(a^2 + bb)}$, and conſequently $x = \sqrt[3]{[aab \pm a^2\sqrt{(a^2 + b^2)}]}$, ſuſceptible of two real Values, one poſitive and the other negative; the other four Roots are imaginary. Third example.

LXXIII.

Let the Equation $x^4 - (aa + bb)xx = -aabb$ be propoſed, adding to both Sides $\frac{1}{4}a^4 + \frac{1}{2}aabb + \frac{1}{4}b^4$, there will reſult $x^4 - (aa + bb)xx + \frac{1}{4}a^4 + \frac{1}{2}aab^2 + \frac{1}{4}b^4 = \frac{1}{4}a^4 - \frac{1}{2}aabb + \frac{1}{4}b^4$, both Members being perfect Squares, extracting therefore the ſquare Root, there will reſult $xx - \frac{1}{2}aa - \frac{1}{2}bb = \pm\frac{1}{2}aa \mp \frac{1}{2}bb$, which gives $xx = aa$ and $xx = bb$, that is, $x = \pm a$ and $x = \pm b$, which are the four Roots of the Equation $x^4 - (aa + bb)xx = -aabb$. Fourth example.

LXXIV.

Let the Equation $x^4 - (2gb + 4ff)xx = -ggbb$ be propoſed to be ſolved, adding to both Sides the Square of half the Coeficient of the ſecond Term, and extracting the ſquare Root of both Members, there reſults $x^2 + gb - 2ff = \pm 2f\sqrt{(gb + ff)}$ which gives $x = \pm\sqrt{[gb + 2ff \pm 2f\sqrt{(gb + ff)}]}$. Fifth example.

Now on Examination I find that this Quantity is a perfect Square, that of $f \pm \sqrt{(ff + gb)}$, for the Term $2f\sqrt{(gb + ff)}$ is double of the Product of $f$ into $\sqrt{(gb + ff)}$, and the Quantity $gb + 2ff$ contains the Square of $f$, and the Square $ff + gb$ of the radical Part.

Whence the foregoing Value of $x$ is reduced to $f \pm \sqrt{(ff + gb)}$ and $-f \pm \sqrt{(ff + gb)}$, and thoſe are the four Roots of the Equation $x^4 - 2gbxx - 4ffxx = -ggbb$.

LXXV.

In Order to diſcover when the Roots of Surds may be expreſſed exactly by other Surds, as in the foregoing Example, and how thoſe Roots may be found, I obſerve that the Root of a Quantity compoſed of two Parts, one of which is rational and the other a radical Quantity of the ſecond Degree, ſhould itſelf conſiſt of two Parts, and that one of them at leaſt ſhould be a radical Quantity.

Let then $A + B$ expreſs in general the propoſed Quantity, $A$ denoting the rational Quantity, and $B$ a radical of the ſecond Degree, and let $p + q$ expreſs the Root ſought.

I now obſerve, that whether $p$ denotes the radical Quantity, or whether it be expreſſed by $q$, or that $p$ and $q$ are both Radicals, in the Square

$p^2 + 2pq + q^2$, the Term $2pq$ can only be a Radical; comparing therefore this Square with the given Quantity, $2pq$ will repreſent $B$ and $p^2+q^2$, $A$; that is, we have the two Equations $p^2 + q^2 = A$ and $2pq = B$ for finding $p$ and $q$. From the ſecond I deduce $p = \frac{B}{2q}$ which ſubſtituted in the firſt gives $\frac{B^2}{4q^2} + q^2 = A$, or $q^4 - Aq^2 = -\frac{B^2}{4}$ or $q^2 - \frac{1}{2}A = \pm\frac{1}{2}\sqrt{(AA - BB)}$, conſequently $q = \pm\sqrt{[\frac{1}{2}A \pm \frac{1}{2}\sqrt{(A^2 - B^2)}]}$. Subſtituting afterwards this Value of $q$ in the Equation $p^2 + q^2 = A$, or $p = \pm\sqrt{(A - q^2)}$ we have $p = \pm\sqrt{[\frac{1}{2}A \mp \frac{1}{2}\sqrt{(A^2 - B^2)}]}$; from whence it follows, that the Root required of the Quantity $A + B$ is $\pm\sqrt{[\frac{1}{2}A \pm \frac{1}{2}\sqrt{(A^2 - B^2)}]} \pm \sqrt{[\frac{1}{2}A \pm \frac{1}{2}\sqrt{(A^2 - B^2)}]}$, or ſimply $\pm\sqrt{[\frac{1}{2}A + \frac{1}{2}\sqrt{(A^2 - B^2)}]} \pm \sqrt{[\frac{1}{2}A - \frac{1}{2}\sqrt{(A^2 - B^2)}]}$.

Method of finding the ſquare root of quantities partly rational and partly irrational.

As to the Signs which ſhould affect the two Parts $\sqrt{[\frac{1}{2}A + \frac{1}{2}\sqrt{(A^2 - B^2)}]}$ and $\sqrt{[\frac{1}{2}A - \frac{1}{2}\sqrt{(A^2 - B^2)}]}$ of the Root required of $A + B$, they are the ſame when the radical $B$ is poſitive, but different when $B$ is negative; for it is eaſy to perceive that in general $p + q$, or $-p - q$ being the Root of $A + B = pp + qq + 2pq$, $p - q$, or $-p + q$ is the Root of $A - B = pp + qq - 2pq$.

LXXVI.

That the Quantity $\pm\sqrt{[\frac{1}{2}A + \frac{1}{2}\sqrt{(A^2 - B^2)}]} \pm \sqrt{[\frac{1}{2}A - \frac{1}{2}\sqrt{(A^2 - B^2)}]}$ found for the Root of the Quantity $A + B$ may be expreſſed by a Binomial Surd, the Quantity $A^2 - B^2$ muſt be a perfect Square; which will never fail to happen as often as the Root of $A + B$ can be extracted: To make this appear we have only to obſerve that $p - q$ is expreſſed by $\sqrt{(A - B)}$ at the ſame Time that $p + q$ is expreſſed by $\sqrt{(A + B)}$; from whence it is eaſy to conclude, that $\sqrt{(A - B)} \times \sqrt{(A + B)}$, or $\sqrt{(A^2 - B^2)} = (p - q) \times (p + q)$, or $p^2 - q^2$ a commenſurable Quantity.

LXXVII.

To ſhew the Application of the foregoing Method, let the Quantity $aa + 2c\sqrt{(aa - cc)}$ be propoſed, comparing it with $A + B$ we have $a^2 = A$ and $B = 2c\sqrt{(a^2 - c^2)}$; and conſequently $\sqrt{(A^2 - B^2)} = aa - 2cc$, whence $\sqrt{[\frac{1}{2}A + \frac{1}{2}\sqrt{(A^2 - B^2)}]} = \sqrt{(a^2 - c^2)}$; in like Manner, $\sqrt{[\frac{1}{2}A - \frac{1}{2}\sqrt{(A^2 - B^2)}]} = c$; that is, the Root ſought is $c + \sqrt{(aa - cc)}$, or $-c - \sqrt{(aa - cc)}$.

Application of the foregoing method to an example.

LXXVIII.

If it was propoſed to extract the ſquare Root of $16 + 6\sqrt{7}$, putting $A = 16$ and $B = 6\sqrt{7}$, there reſults $\sqrt{(AA - BB)} = 2$, and conſequently $\sqrt{[\frac{1}{2}A + \frac{1}{2}\sqrt{(AA - BB)}]} = 3$, and $\sqrt{[\frac{1}{2}A - \frac{1}{2}\sqrt{(AA - BB)}]} = \sqrt{7}$, whence $3 + \sqrt{7}$, or $-3 - \sqrt{7}$ will be the Root required.

LXXIX.

Another example.

Let the Root of $a\,p - 2\,a\,\sqrt{(a\,p - a\,a)}$ be required, putting $A = a\,p$ and $B = -2\,a\,\sqrt{(a\,p - a\,a)}$, there reſults $\sqrt{(A\,A - B\,B)} = a\,p - 2\,a\,a$ and $\sqrt{[\frac{1}{2}A + \frac{1}{2}\sqrt{(A\,A - B\,B)}]} = \sqrt{(a\,p - a\,a)}$: In like Manner, $\sqrt{[\frac{1}{2}A - \frac{1}{2}\sqrt{(A^2 - B^2)}]} = a$; that is, the Root ſought will be $\sqrt{(a\,p - a\,a)} - a$, or $a - \sqrt{(a\,p - a\,a)}$.

LXXX.

Third example.

If the Quantity $b^2 - a\,b + \frac{1}{4}a^2 + 2\sqrt{(a\,b^3 - 2\,a^2\,b^2 + \frac{1}{4}a^3\,b)}$ was propoſed, then $A = b^2 - a\,b + \frac{1}{4}a\,a$, $B = 2\sqrt{a\,b^3 - 2\,a^2\,b^2 + \frac{1}{4}a^3\,b}$, and $\sqrt{(A^2 - B^2)} = \sqrt{b^4 - 6\,a\,b^3 + \frac{19}{2}a^2\,b^2 - \frac{3}{2}a^3\,b + \frac{1}{16}a^4}$ $= b\,b - 3\,a\,b + \frac{1}{4}a\,a$, which will give $\sqrt{[\frac{1}{2}A + \frac{1}{2}\sqrt{(A^2 - B^2)}]}$ $= \sqrt{(b\,b - 2\,a\,b + \frac{1}{4}a\,a)}$, and $\sqrt{[\frac{1}{2}A - \frac{1}{2}\sqrt{(A^2 - B^2)}]} = \sqrt{a\,b}$; whence the Root ſought is $\sqrt{a\,b} + \sqrt{(b\,b - 2\,a\,b + \frac{1}{4}a\,a)}$, or $-\sqrt{a\,b} - \sqrt{(b\,b - 2\,a\,b + \frac{1}{4}a\,a)}$.

LXXXI.

Method of finding the cube root of quantities, partly rational and partly irrational.

After the Analyſts had found out a Method for diſcovering among the Quantities partly rational and partly irrational ſuch as were perfect Squares, they, without Doubt, ſought alſo how to diſtinguiſh thoſe that were perfect Cubes or higher Powers, as being abſolutely neceſſary to compleat the Reſolution of Equations contained in the Formula $x^{2m} + a\,x^m = b$, or $x = \sqrt[m]{[-\frac{1}{2}a \pm \sqrt{(b + \frac{1}{4}a\,a)}]}$. Let us firſt examine what muſt be done when $m = 3$, that is, to find the cube Root of any Quantity $A + B$ in which $A$ is rational, and $B$ a Radical of the ſecond Degree.

I obſerve firſt, that the cube Root of a Quantity of this Nature cannot include more than one Radical of the ſecond Degree, for it is eaſy to perceive, that the Cube of a Quanty ſuch as $\sqrt{m} + \sqrt{n}$ which contains two of thoſe Radicals, will neceſſarily be affected by thoſe ſame Radicals.

I obſerve next, that the ſame cube Root ſought cannot contain any other kind of Radicals except a cubical Radical, and which is common to the two Parts of the Root. Such would be, for Example, the Quantity of $f\sqrt[3]{m} + \sqrt{g} \times \sqrt[3]{m}$, whoſe Cube $m\,f^3 + 3\,m\,f\,g + (3\,m\,f\,f + m\,g)\sqrt{g}$, as the propoſed Quantity $A + B$ conſiſts of a commenſurable Quantity, and a radical Part of the ſecond Degree.

Now let $p + q$ expreſs the Root ſought $q$ being the Part affected by the Radical of the ſecond Degree, whether it be affected as well as $p$ by a cubical Radical or not.

It is manifeſt that the Cube $p^3 + 3\,p^2\,q + 3\,p\,q^2 + q^3$ will not contain any other Radical, but the Radical of the ſecond Degree, which is in $q$, and that only the Terms $3\,p\,p\,q$ and $q^3$ will be affected by this Radical; I therefore compare thoſe two Terms with the given Quantity $B$ and the two others with $A$, which gives the Equations $A = p^3 + 3\,p\,q^2$ and $B = 3\,p^2\,q + q^3$.

To ſolve thoſe two Equations we muſt firſt exterminate one of the unknown Quantities $p$ or $q$, which may be effected by any of the Methods explained in the ſecond Chapter, but much eaſier by obſerving, that if $p+q$ is the cube Root of $p^3+3p^2q+3pq^2+q^3$, the firſt Part of which $p^3+3pq^2=A$, and the ſecond $3p^2q+q^3=B$, $p-q$ will neceſſarily be the cube Root of $A-B$; which is in this Caſe $=p^3+3pq^2-3p^2q-q^3$, whence $A+B=(p+q)^3$ and $A-B=(p-q)^3$; conſequently, $(A+B)(A-B)=(p+q)^3(p-q)^3$, or $AA-BB=(pp-qq)^3$ or $\sqrt[3]{(AA-BB}=p^2-q^2$, or $n=p^2-q^2$, in which $n$ is given, and muſt be either a commenſurable Quantity, or a ſimple cubical Radical. From the Equation $n=p^2-q^2$, or $q^2=p^2-n$, and from the Equation $A=p^3+3pq^2$ is deduced $4p^3-3pn-A=0$, by Means of which Equation the Part $p$ of the cubical Root ſought may be obtained, and when found, the ſecond Part of the Root ſought will be obtained by Means of the Equation $q=\sqrt{(p^2-n)}$.

As to the radical Sign it will be poſitive, if the Radical of the propoſed Quantity has the Sign $+$, and negative, if the Radical of the propoſed Quantity has the Sign $-$; for it is eaſy to perceive, that the radical Part of the Cube of $p+q$ which is $(3pp+qq)\times q$ will always have the ſame Sign as $q$.

If the Root of the propoſed Quantity $A+B$ does not contain a cubical Radical affecting all its Terms, $n$ or $\sqrt[3]{(AA-BB)}$ will be a commenſurable Quantity, and conſequently, the Equation $4p^3-3pn-A=0$ will not be affected by any radical Quantity, and as $p$ in this Caſe will be rational, we cannot fail of finding it by inveſtigating all the Diviſors of this Equation.

If the Root ſought ſhould have its two Parts affected by a cubical Radical, which will happen as often as $AA-BB$ is not a perfect Cube, then $AA-BB$ muſt be multiplied by a Quantity $Q$, ſuch as that the Product $A^2Q-B^2Q$ may be a perfect Cube; and now inſtead of $A+B$ we muſt extract the cube Root of $(A+B)\times\sqrt{Q}$, which being found and divided by the cube Root of $\sqrt{Q}$ will give the Root required.

LXXXII.

Application of the foregoing method to an example.

To ſhew the Application of this Method, let it be required to find the cube Root of the Quantity $7a^3-3a^2b+(5aa-ab)\times\sqrt{(2aa-ab)}$.

Comparing this Quantity with $A+B$ I find $7a^3-3a^2b=A$, $[5aa-ab\sqrt{(2aa-ab)}]=B$; and conſequently, $A^2-B^2=-a^6+3a^5b-3a^4b^2+a^3b^3$, and $n$ or $\sqrt[3]{(A^2-B^2)}=-aa+ab$, ſubſtituting this Value of $n$, as alſo that of $A$ in the Equation $4p^3-3pn-A=0$, there reſults $4p^3+3paa-3pab=7a^3+3a^2b$, which is diviſible by $p-a$, that is, the Value of $p$ is $a$. Subſtituting this Value of $p$ in the Equation $q=\sqrt{(pp-n)}$, there reſults $q=\sqrt{(2aa-ab)}$, whence the Root ſought is $a+\sqrt{(2aa-ab)}$.

LXXXIII.

Let it be required to find the cube Root of $2aac - abc - bbc - [2a - b \sqrt{(aacc - bbcc)}]$, putting $2aac - abc - bbc = A$ and $B = -[2a - b \sqrt{(aacc - bbcc)}]$, there results $A^2 - B^2 = 2b^4cc - 2ab^3cc$, which is not a perfect Cube. To discover what will make it a perfect Cube, I resolve it into its Factors, and it becomes $2 \times cc \times (b - a) \times b^3$; whence it appears that by multiplying by $\frac{4 \times (b-a)^2}{cc}$ it will become a perfect Cube, that of $2 \times (b - a) \times b$, or of $2bb - 2ab$; and consequently if the proposed Quantity be multiplied by $\frac{2b-2a}{c}$ square Root of $\frac{4 \times (b-a)^2}{cc}$, there will result a new Quantity $[(2aa - ab - bb) \times (2b - 2a) - (2a - b) \times (2b - 2a) \times \sqrt{(aa - bb)}]$, the first Part of which representing $A$, and the second $B$, will give for $\sqrt[3]{(AA - BB)}$, that is, for $n$, $2bb - 2ab$. Whence the Equation $4p^3 - 3pn - A = 0$ will be transformed into $4p^3 - 3p \times (2bb - 2ab) + (bb + ab - 2aa) \times (2b - 2a) = 0$, which is divisible by $p + a - b$, that is, $p = b - a$, which Value being substituted in $q = \sqrt{(pp - n)}$ there results $q = \sqrt{(aa - bb)}$; wherefore $b - a - \sqrt{(aa - bb)}$ will be the cube Root of the Product of the proposed Quantity into $\frac{2b-2a}{c}$; wherefore $\frac{[b-a-\sqrt{(a^2-bb)}] \times \sqrt[3]{c}}{\sqrt[3]{(2b-2a)}}$ is the cube Root required of the proposed Quantity.

Another example.

LXXXIV.

If the Terms of the Quantity whose cube Root is to be extracted have Divisors, they are first to be reduced to the same Denominator, and then the cube Root of the Numerator is to be divided by that of the Denominator.

LXXXV.

If it be proposed to extract the Root of a numerical Quantity partly rational and partly irrational, it may be performed much easier than by the foregoing Method.

Method of finding the cube roots of numeral quantities, partly rational and partly irrational.

For supposing first, that the Root sought should not be affected by a cubical Radical, but should consist of an Integer, and a radical Part, also an Integer Number, because $p - q = \sqrt[3]{(A - B)}$ when $p + q = \sqrt[3]{A + B}$ we will have $p = \frac{\sqrt[3]{(A-B)} + \sqrt[3]{(A+B)}}{2}$; whence the rational Part of the Root will be obtained by computing in the neareſt integer Numbers, $\sqrt[3]{(A - B)}$ and $\sqrt[3]{(A + B)}$, and taking the one half of these two Numbers. For assuming for $\sqrt[3]{(A - B)}$ and for $\sqrt[3]{(A + B)}$ the neareſt integer Numbers, neglecting the fractional Parts, the Error that can arise in each of these Quantities will be less than $\frac{1}{2}$, and consequently the Integer Number that results for the Value of $\frac{\sqrt[3]{(A-B)} + \sqrt[3]{A+B}}{2}$, that is, for $p$, will not

differ from its true Value by Unity, and as this Value of $p$ ſhould be an Integer, it will be exactly determined by this Means.

Having thus obtained the Value of $p$, and that of $n$ or of $\sqrt[3]{(AA-BB)}$ being given, ſubſtituting the Values of $p$ and $n$ in $q=\sqrt{(pp-n)}$, the ſecond Part of the cube Root required will be obtained.

If the Root ſought ſhould have its two Terms affected by a cubical Radical, which happens as often as $A^2-B^2$ is not a perfect Cube, then as in ſpecious Quantities we muſt extract the cube Root of $A\sqrt{Q}+B\sqrt{Q}$, $Q$ being a Number ſuch as that $A^2Q-B^2Q$ may be a perfect Cube, and having found this Root, divide it by the cube Root of $\sqrt{Q}$.

LXXXVI.

To ſhew the Application of this Method, let the Cube Root of $7+5\sqrt{2}$ be required, putting $A=7$, $B=5\sqrt{2}$, I find $n$ or $\sqrt[3]{(A^2-B^2)}=-1$, I afterwards obſerve that the Value of $\sqrt[3]{(A+B)}$, or of $\sqrt[3]{(7+5\sqrt{2})}$ is nearer 2 than 3. I therefore aſſume 2 for its Value, in like Manner I obſerve that $\sqrt[3]{(A-B)}$, or $\sqrt[3]{(7-5\sqrt{2})}$ is nearer 0 than 1, I aſſume 0 for this Quantity, whence $p$, or $\frac{\sqrt[3]{(A+B)}+\sqrt[3]{(A-B)}}{2}=1$. I ſubſtitute this Value of $p$ in $q=\sqrt{(pp-n)}$, and I find $q=\sqrt{2}$, whence I conclude, that if the cube Root of the propoſed Quantity $7+5\sqrt{2}$ can be extracted, it will be $1+\sqrt{2}$, and in Effect, $1+\sqrt{2}$ raiſed to the Cube, gives $7+5\sqrt{2}$.

Application of the foregoing method to an example.

LXXXVII.

Let the cube Root of $5+3\sqrt{3}$ be required. Here $AA-BB=-2$, but 2 not being a perfect Cube, I examine what Number $\sqrt{Q}$, by which $5+3\sqrt{3}$ being multiplied, will render $AA-BB$ a perfect Cube, and finding 2 for this Number, I inveſtigate the cube Root of $10+6\sqrt{3}$, and I find $n=-2$, and the neareſt Value of $\frac{\sqrt[3]{(A+B)}+\sqrt[3]{(A-B)}}{2}$ or of $\frac{\sqrt[3]{(10+6\sqrt{3})}+\sqrt[3]{(10-6\sqrt{3})}}{2}$ in Integers to be 1, ſubſtituting thoſe Values of $p$ and $n$ in $q=\sqrt{(pp-n)}$ I find $q=\sqrt{3}$; and finding upon Examination that $p+q$ or $1+\sqrt{3}$ is the cube Root of $10+6\sqrt{3}$, I conclude that $\frac{1+\sqrt{3}}{\sqrt[3]{2}}$ is the cube Root ſought of $5+3\sqrt{3}$.

Another example.

LXXXVIII.

The Computation for determining $p$ may be rendered more ſimple by obſerving that inſtead of $\frac{\sqrt[3]{(A+B)}+\sqrt[3]{(A-B)}}{2}$ we may put $\frac{\sqrt[3]{(A+B)}+\frac{n}{\sqrt[3]{(A+B)}}}{2}$, Since $n$ or $\sqrt[3]{(A^2-B^2)}=$

The foregoing method rendered more ſimple.

$\sqrt[3]{(A+B)} \times \sqrt[3]{(A-B)}$. Which is ſimpler than $\frac{\sqrt[3]{(A+B)}+\sqrt[3]{(A-B)}}{2}$ becauſe it is eaſier to divide the Number $n$ already found by $\sqrt[3]{(A+B)}$ than to compute ſeparately $\sqrt[3]{(A-B)}$.

LXXXIX.

To ſhew the Application of this new Formula, let it be applied to the Example of Art. LXXXV. where $A$ was $= 7$, and $B = 5\sqrt{2}$. After having found, as in the ſame Article, that $n = -1$, and that $\sqrt[3]{(A+B)}$ in the neareſt integer Number, to be 2, inſtead of ſearching, as in the ſame Article, the cube Root of $7 - 5\sqrt{2}$ in the neareſt integer Number, I divide $n$ or $-1$ by the Value 2 of $\sqrt[3]{(A+B)}$ which gives $-\frac{1}{2}$ which being ſubſtituted in the foregoing Formula, which expreſſes the Value of $p$, $\frac{\sqrt[3]{A+B}+\frac{n}{\sqrt[3]{(A+B)}}}{2}$ there reſults $\frac{2-\frac{1}{2}}{2}$, or 1 (aſſuming the neareſt integer Number) for the Value of $p$, as found in the above-mentioned Article by the Formula $\frac{\sqrt[3]{(A+B)}+\sqrt[3]{(A-B)}}{2}$, the Remainder of the Operation is performed as in the ſame Article. Application of the new method.

XC.

It is to be obſerved however, that the new Formula may lead into Error, if $A$ and $B$ have not the ſame Sign, for when theſe Quantities have different Signs, the real Value of $\sqrt[3]{(A+B)}$ may be ſo ſmall with reſpect to $n$, that the neareſt integer Number aſſumed for this Value, will give for $\frac{\sqrt[3]{(+B)}+\frac{n}{\sqrt[3]{(A+B)}}}{2}$ a Number that will differ from its true Value by ſeveral Units. For Example, if it was required to extract the cube Root of $45 - 29\sqrt{2}$, putting $A = 45$ and $B = -29\sqrt{2}$, the Value of $\sqrt[3]{(A+B)}$ or of $\sqrt[3]{(45-29\sqrt{2})}$ in the neareſt integer Number will be 1, and as $\sqrt[3]{(A^2-B^2)}$ is here equal to 7, the Value of $p$, or of $\frac{\sqrt[3]{(A-B)}+\frac{n}{\sqrt[3]{(A+B)}}}{2}$ will be found in this Caſe to be 4, which in Reality is only equal to 3, as may be proved by the Expreſſion $\frac{\sqrt[3]{(A+B)}+\sqrt[3]{(A-B)}}{2}$ of the foregoing Method. This new method defective when $A$ and $B$ have different ſigns.

But provided that this new Method of finding the Value of $p$ is exact, when $A$ and $B$ have the ſame Sign, it is of little Conſequence whether it be applicable or not, when thoſe Quantities have different Signs. For it is eaſy to perceive that in this Caſe, we have only to ſuppoſe $A$ and $B$ to be

both positive, and extract the Root $p + q$, then give $p$ the same Sign that $A$ has, and $q$ the same Sign that $B$ has.

It remains then to examine, whether when $A$ and $B$ have the same Sign, or what amounts to the same Thing, if $A$ and $B$ be both positive, we may substitute in $\frac{\sqrt[3]{(A+B)} + \frac{n}{\sqrt[3]{(A+B}}}{2}$ in the Room of $\sqrt[3]{(A+B)}$ its Value in the nearest integer Number, so that the Value which will result will not differ from its true Value by Unity, to be assured of it, let the Value of $\sqrt[3]{(A+B)}$ in the nearest integer Number be supposed to differ from its true Value by $\frac{1}{2}$, which is more than can ever happen. In this Case, the Value of $p$ will be $\frac{\sqrt[3]{(A+B)} \pm \frac{1}{2} + \frac{n}{\sqrt[3]{(A+B)} \pm \frac{1}{2}}}{2}$ to shew that this Expression of $p$ can not differ from it's true Value by Unity, let $p + q$ be substituted in the Room of $\sqrt[3]{(A+B)}$ and $pp + qq$ in the Room of $n$, and there will result $\frac{p+q \pm \frac{1}{2} + \frac{pp+qq}{p+q \pm \frac{1}{2}}}{2}$ from which deducting $p$, the Remainder $\frac{\pm q + \frac{1}{4}}{2p+2q \pm 1}$ will express the difference between the Value of $p$ in the nearest Integer, and its true Value, now, it is manifest, that this Quantity can never equal $\frac{1}{2}$, for in the first Expression $\frac{q + \frac{1}{4}}{2p+2q+1}$ that it includes, the Numerator $q + \frac{1}{4}$ being less than the one half of $2q + 1$, will be much less than the one half of $2p + 2q + 1$: and in the second Expression $\frac{-q + \frac{1}{4}}{2p+2q-1}$ which it likewise includes, the Numerator $-q + \frac{1}{4}$, being less than the one Half of $2q$, will be also less than the one half of $2q + 2p - 1$.

What is to be done in this case.

Whence the Value of $p$ determined by the foregoing Method, cannot differ from its true Value by Unity, and consequently as often as a Quantity $A + B$, partly rational and partly irrational, (in which only Integers enter either under the radical Sign, or before this Sign) will have a cube Root $p + q$, expressed in Integers, this Root may be found by the foregoing Method.

XCI.

But if the Quantity $A + B$, though clear of Fractions, shall have a fractional Root, such as the Quantity $2 + \sqrt{5}$, whose cube Root is $\frac{1}{2} + \frac{1}{2}\sqrt{5}$, it cannot be found by either of the two foregoing Methods.

Case in which the two foregoing methods fail.

To remedy this Inconveniency, the Analysts have sought in a direct Manner all the fractional Roots, which when cubed, become integer Numbers.

Let all those Roots be expressed by $\frac{p}{m} + \frac{q}{n}$; $p$ and $m$ expressing Integers that have no common Divisor, and $q$ the Root of an integer Number, which will admit of no Reduction with the Number $n$. Raising this Quantity to the Cube, there will result $\frac{p^3}{m^3} + \frac{3pq^2}{mn^2}$ for the rational Part, and $\left[\frac{3p^2}{nm^2} + \frac{q^2}{n^3}\right] \times q$ for the irrational Part. Now, by the Conditions of the Problem, the first Part $\frac{p^3}{m^3} + \frac{3pq^2}{mn^2}$, and the Coefficient $\frac{3pp}{nm^2} + \frac{qq}{n^3}$ of the second Part should be integer Numbers.

How to remedy this deficiency.

Let $\frac{3p^2}{m^2n} + \frac{q^2}{n^3}$ be put equal to $b$, which will express an integer Number. Then $q^2 = bn^3 - \frac{3p^2n^2}{m^2}$; but this Quantity by Hypothesis should be an integer Number; consequently $\frac{3p^2n^2}{m^2}$ should be also an Integer; wherefore $\frac{3n^2}{m^2}$ should be likewise an Integer, and consequently $n$ should be a Multiple of $m$.

Now let $n = ml$, consequently $q^2 = bm^3l^3 - 3p^2l^2$, substituting those Values of $n$ and $q^2$ in $\frac{p^3}{m^3} + \frac{3pq^2}{mn^2}$, this Quantity will become after the Reductions $-\frac{8p^3}{m^3} + 3pbl$, which should be an Integer, but $p$ and $m$ having no common Divisor, this Quantity cannot be an integer Number, unless $m$ be 1 or 2. As to $n$ or $ml$, it is easy to perceive that it should be equal to $m$, because the Equation $q^2 = bm^3l^3 - 3p^2l^2$ will give $\frac{q}{l} = \sqrt{(bm^3l - 3p^2)}$ and consequently $\frac{q}{ml}$, that is, $\frac{q}{n} = \frac{1}{m}\sqrt{(bm^3l - 3p^2)}$ whence it appears that the second Part of the Root can have no other Denominator than what the first has, and consequently must be either 2 or 1.

When therefore the Root of a Quantity $A + B$, the rational Part of which $A$, and the irrational Part $B$ is expressed in Integers, cannot be found by the foregoing Method, we have no more to do than to muliply this Quantity by 8, and investigate by the foregoing Method the Cube Root of the Quantity resulting from this Multiplication, and if this does not succeed, no farther Trial is to be made: if it succeeds, the one Half of the cube Root found, will be the Root required.

XCII.

When the Number $A$ and the Radical $B$ are Fractions, they are to be reduced to the ſame Denominator, and the Root of the Numerator and Denominator are to be extracted ſeparately. Thus, to extract the cube Root of $\sqrt{242 - 12\frac{1}{2}}$, this reduced to a common Denominator is $\frac{\sqrt{968 - 25}}{2}$, and the Roots of the Numerator and Denominator found ſeparately give the Root $\frac{2\sqrt{2} - 1}{\sqrt[3]{2}}$. And if you are to extract the cube Root out of $\sqrt[3]{3993 + \sqrt[6]{17578125}}$, divide its Parts by the common Diviſor $\sqrt[3]{3}$, and the Quotient being $11 + \sqrt{125}$, the cube Root of the propoſed Quantity will be found by taking the cube Roots of $\sqrt[3]{3}$, and of $11 + \sqrt{125}$, and multiplying them into each other.

XCIII.

What is to be done when the cube root ſhould be the Sum of two radicals.

If it be propoſed to extract the cube Root of a Quantity compoſed of two Radicals of the ſecond Degree, whether it be a ſpecious or numeral Quantity, we have no more to do than to multiply it by the Cube of one of the Radicals that it contains, the Product that reſults being a Quantity partly rational and partly irrational. Its Root may be extracted by the foregoing Method, which when found is to be divided by the Radical, by the Cube of which the propoſed Quantity was multiplied.

XCIV.

How the root denominated by an even number is extracted.

If the fourth Root of a Quantity as $A + B$ was required; firſt, the ſquare Root is to be extracted, which, if it cannot be found, much leſs can the Root of the fourth Power be found. In like Manner, all even Roots are to be ſought, continually depreſſing them by extracting the ſquare Root.

XCV.

In general, if the $m^{th}$ Root of the Quantity $A + B$ was required, then $(p + q)^m = A + B$, and raiſing $p + q$ to the Power $m$, it will appear that $A$ is the Sum of all the odd Terms $p^m$, $\frac{m \cdot (m-1)}{2} q^2 p^{m-2}$, $\frac{m(m-1)(m-2)(m-3)}{2 \times 4} q^4 p^{m-4}$, &c. and that $B$ is the Sum of all the even Terms $m q p^{m-1}$, $\frac{m(m-1)(m-2)}{2 \times 3} q^3 p^{m-3}$, $\frac{m(m-1)(m-2)(m-3)(m-4)}{2 \times 3 \times 4 \times 5} q^5 p^{m-5}$, &c. whence, when $A+B$ is the $m^{th}$ Power of $p + q$, $A - B$ will be the $m^{th}$ Power of $p - q$; conſequently $\sqrt[m]{(A-B)} \times \sqrt[m]{(A+B)} = (p+q) \times (p-q)$, or $pp - qq = n$, or $q = \sqrt{p^2 - n}$, alſo $\frac{\sqrt[m]{(A+B)} + \sqrt[m]{(A-B)}}{2}$

Inveſtigat. of the rule for extracting the root of any power of quantities partly rational and partly irrational.

$= \frac{p+q+p-q}{2}$, or $p = \frac{\sqrt[m]{(A+B)} + \frac{n}{\sqrt[m]{(A+B)}}}{2}$, because $\sqrt[m]{(AA-BB)} = \sqrt[m]{(A+B)} \times \sqrt[m]{(A-B)}$, and thus from the Compoſition of the Binomial $A+B$ we are lead to its Reſolution, when $A$ is rational, and $A^2 - B^2$ is a perfect $m$ Power.

XCVI.

If the $m^{th}$ Root of $A^2 - B^2$ cannot be taken, multiply $A^2 - B^2$ by a Number $\mathcal{Q}$, ſuch as that the Product may be the leaſt perfect $m$ Power $x^m = A^2\mathcal{Q} - B^2\mathcal{Q}$; and now inſtead of $A+B$ extract the $m$ Root of $(A+B) \times \mathcal{Q}$, as above, and when found, divide it by $\sqrt[m]{\mathcal{Q}}$, and the Quotient will be the Root required.

To find this leaſt perfect Power $m$ ($n^m$) that ſhall be a Multiple of $A^2 - B^2$ by a whole Number $\mathcal{Q}$, let this given Number $A^2 - B^2$ be repreſented by the Product $a^a\, b^b\, d\, f$, whoſe ſingle Diviſors let be $a\,a\,a\,\ldots\, b\,b\,b\,\ldots\,d, f$, and the Product of theſe Diviſors raiſed to the Power $m$, which is $a^m\, b^m\, d^m\, f^m$ divided by $a^a\, b^b\, d\, f$, will give the Quotient $a^{m-a}\, b^{m-b}\, d^{m-1}\, f^{m-1} = \mathcal{Q}$ a whole Number, provided ſome Index as $n$ or $p$ be not greater than $m$; if it is, take, inſtead of the ſingle Diviſor, $a$ or $b$, $a^2$ or $b^2$, $a^3$ or $b^3$, &c. till there be no negative Index in the Quotient, that is, till $\mathcal{Q}$ be a whole Number.

XCVII.

Reſolution of equations of even dimenſions by compleating them into perfect ſquares explained by an example.

Having ſhewn how Equations are ſolved by compleating them into perfect Squares, the Square on the right Hand Side being a Quantity entirely known, we ſhall now proceed to explain how Equations are compleated into perfect Squares, the Squares on both Sides being affected by the unknown Quantity. Let the Equation $x^4 - 9x^3 + 15x^2 - 27x + 9 = 0$; be propoſed to be ſolved according to the Rules of the Extraction of the ſquare Root, I find the two firſt Terms of the Root of this Quantity to be $x^2 - \frac{9}{2}x$. Let the whole Root be expreſſed by $x^2 - \frac{9}{2}x + A$; now obſerving that the laſt Term 9 of the propoſed Equation is a perfect Square, as likewiſe the laſt Term of the aſſumed Equation $(x^2 - \frac{9}{2}x + A)^2 = x^4 - 9x^3 + \left(2A + \frac{81}{4}\right)x^2 - 9Ax + A^2$. I put $A = 3$ which will transform this Equation into $x^4 - 9x^3 + \frac{105}{4}x^2 - 27x + 9$, whoſe Terms agree with all the Terms of the propoſed Equation, except the third; the Difference of thoſe Terms being $\frac{45}{2}x^2$; to compleat therefore the propoſed Equation into a

perfect Square, add to both Sides of it $\frac{45}{2} x^2$, and extracting the Square Root of it, I find $x^2 - \frac{9}{2} x + 3 = \frac{3}{2} x \sqrt{5}$, and consequently, $x = \frac{9 \pm 3}{4} \sqrt{5} \pm \sqrt{\left[\frac{39}{4} \pm \frac{27}{8} \sqrt{5}\right]}$.

XCVIII.

Another example.

Let the Equation $x^4 - 2ax^3 + (2a^2 - cc)x^2 - 2a^3x + a^4 = 0$ be proposed to be solved, according to the Rules of the Extraction of Roots I find the two first Terms of the Root of this Quantity to be $x^2 - ax$: Let the whole Root be expressed by $x^2 - ax + A$; now observing that the last Term $a^4$ of the proposed Equation to be a perfect Square, as likewise the last Term of the assumed Equation $(x^2 - ax + A)^2 = x^4 - 2ax^3 + (2A + a^2)x^2 - 2aAx + A^2$: I put $A = a^2$, which will transform this Equation into $x^4 - 2ax^3 + 3a^2x^2 - 2a^3x + a^4$, whose Terms agree with the Terms of the proposed Equation, except the third, the Difference of those Terms being $(3a^2 - 2a^2 + c^2)x^2 = (a^2 + c^2)x^2$; to compleat therefore the proposed Equation into a perfect Square, I add to both Sides of it $(a^2 + c^2)x^2$, and extracting the square Root of it, I find $x^2 - ax + a^2 = x\sqrt{(a^2 + c^2)}$, and consequently $x = \frac{1}{2}a \pm \frac{1}{2}\sqrt{(aa + cc)} \pm \sqrt{[(\frac{1}{4}cc - \frac{1}{2}aa \pm \frac{1}{2}a)\sqrt{(aa + cc)}]}$.

XCIX.

Investigation of the method of compleating equations of four dimensions into perfect squares.

Let in general the Equation $x^4 + px^3 + qx^2 + rx + s = 0$, be proposed to be solved, in which $p$, $q$, $r$, $s$, are the given Coeficients, and the Equation is clear of Fractions and Surds;

$$\begin{array}{l|l}
x^4 + px^3 + qx^2 + rx + s & \left\{\begin{array}{l} 2x^2 + px + \mathfrak{Q} \\ x^2 + \frac{1}{2}px + \mathfrak{Q} \\ 2x^2 + \frac{1}{2}px \end{array}\right. \\
-x^4 & \\
\hline
+px^3 + qx^2 + rx + s & \\
-px^3 - \frac{1}{4}p^2x^2 & \\
\hline
+(q - \frac{1}{4}p^2)x^2 + rx + s & \\
+A^2x^2 + 2ABx + B^2 & \\
-2\mathfrak{Q}x^2 - \mathfrak{Q}px - \mathfrak{Q}^2 & \\
\hline
0 \quad 0 \quad 0 &
\end{array}$$

According to the Rules of the Extraction of Roots I find the two first Terms of the Root of this Quantity to be $x^2 - \frac{1}{2}px$. Let the Square $AAx^2 + 2ABx + B^2$ compleat it into a perfect Square; so that the whole Root shall be expressed by $x^2 + \frac{1}{2}px + \mathfrak{Q}$.

I observe that $-\frac{1}{4}p^2 + A^2$, if they are Fractions they should have the same Denominator, since they destroy each other, or their Result is an Integer; consequently $\mathfrak{Q}$ is an Integer, or half of an Integer.

Now from the Equations $q - \frac{1}{4}p^2 + A^2 - 2\mathcal{Q} = 0$, $r + 2AB - p\mathcal{Q} = 0$, and $s + B^2 - \mathcal{Q}^2 = 0$, putting $q - \frac{1}{4}pp = \alpha$ we have $2\mathcal{Q} = A^2 + \alpha$, $p\mathcal{Q} = 2AB + r$, $\mathcal{Q}^2 = B^2 + s$; ſubſtituting therefore the Value of $\mathcal{Q}$, as given by the firſt Equation, in the other two, we ſhall get $\frac{1}{2}pA^2 - 2AB = \beta$, and $\frac{1}{4}A^4 + \frac{1}{2}\alpha A^2 - B^2 = \zeta$: Suppoſing $\beta = r - \frac{1}{2}\alpha p$, and $\zeta = s - \frac{1}{4}\alpha\alpha$. In which Equations the unknown Quantities appertaining to the latter of the two aſſumed Squares are only concerned, and from which their Values might be found; but as the reſulting Equation, when one of the Quantities is exterminated, riſes to the ſixth Dimenſion, and would require more Trouble to reduce it than even the original one propounded, little Advantage would be reaped there from: The Analyſts therefore, inſtead of proceeding in a direct Manner, have endeavoured to diſcover ſuch Properties or Relations of thoſe Quantities as might enable them to gueſs at their Values, which may be afterwards tried by Means of the Equations here exhibited.

The reduction of equations by compleating them into perfect ſquares cannot otherwiſe be effected than by trial.

C.

To explain what they have imagined in this Reſpect, it is to be obſerved, that $A$ and $B$ may be either rational Quantities or Radicals, and in this laſt Caſe, as $A^2x^2 + 2ABx + B^2$ ſhould be a rational Quantity, $A$ and $B$ muſt be affected by the ſame Radical. Let then $A = k\sqrt{n}$, and $B = l\sqrt{n}$, by this Means the two Equations derived above will be changed to $\frac{1}{2}pk^2n - 2kln = \beta$, and $\frac{1}{4}k^4n^2 + \frac{1}{2}\alpha k^2n - l^2n = \zeta$, or to $\frac{1}{2}pk^2 - 2kl = \frac{\beta}{n}$, and $\frac{1}{2}k^4n + \alpha k^2 - 2l^2 = \frac{2\zeta}{n}$ reſpectively.

Now ſince $k$ and $l$ are Integers, or Halves of Integers, conſequently $n$ an Integer, it is plain, that $\frac{\beta}{n}$ and $\frac{2\zeta}{n}$ muſt be Integers likewiſe, or at leaſt the Halves of Integers, and conſequently that $n$ (whoſe Value we are ſeeking) ought to be ſome common integral Diviſor of $\beta$ and $2\zeta$.

How equations of four dimenſions may be reduced in this manner

Moreover, with Regard to $k$ and $l$, it is evident from the firſt of thoſe Equations $\frac{1}{2}pk - 2l = \frac{\frac{\beta}{n}}{k}$ that the former $k$ ought to be ſome Diviſor of $\frac{\beta}{n}$, and that if the Quotient $\frac{\frac{\beta}{n}}{k}$ be taken from $\frac{1}{2}pk$ the Remainder will be the Double of $l$.

It further appears, from the Equations $\mathcal{Q} = \frac{A^2 + \alpha}{2}$, and $\mathcal{Q}^2 = B^2 + s$, by ſubſtituting for $A^2$ and $B^2$ their Equals $nk^2$ and $nl^2$, that $\mathcal{Q}$ will be

$= \frac{a + n k^2}{2}$, and $l^2 = \frac{\mathcal{Q}\mathcal{Q} - s}{n}$, from the former of which $\mathcal{Q}$ will be known, when $n$ and $k$ are known; by Means whereof and the other Equation, $l$ may be a ſecond Time found, and the Agreement or Coincidence of this Value, with that before determined for $l$, will be a Proof that $n$ and $k$ have been rightly aſſumed, and that adding to the given Equation the Quantity $n \times (k x + l)^2$ it will be compleated into the Square $(x^2 + \frac{1}{2} p x + \mathcal{Q})^2$.

CI.

The number of trials diminiſhed from the conſideration of even and odd numbers.

Since the foregoing Method of Solution depends upon the aſſuming proper Diviſors of $\beta$, $2\zeta$ and $\frac{\beta}{n}$ for the Values of $n$ and $k$, it will be expedient, in Order to diminiſh the Number of Trials, to diſcover ſuch of the Diviſors as are not to the Purpoſe, which may be effected from the Conſideration of the Properties of even and odd Numbers.

In Order to which $\mathcal{Q} = \frac{a + n k^2}{2}$ being previouſly transformed to $n k^2 = 2\mathcal{Q} - a = 2\mathcal{Q} - q + \frac{1}{2} p]^2 = \frac{1}{2} p]^2 - f$ (by putting $q - 2\mathcal{Q} = f$). It is evident from thence, that if $p$ be an odd Number $p^2 - 4f$, and conſequently its Equal $4 n k^2$, will likewiſe be an odd Number, becauſe an even Number $4f$, ſubtracted from the Square of an odd one, always leaves odd; therefore ſeeing $4 k^2 \times n$ is here an odd Number, both $n$ and $4 k^2$ muſt be odd, (for the Product of two even Numbers, or of an odd one and an even one is even and not odd); whence it follows, becauſe $2 k]^2$ is odd, that $2 k$ muſt be odd too; and conſequently $k$ the Half of an odd Number.

Now ſeeing $p$, $n$, and $2 k$ are all of them odd Numbers (when $p$ is ſuch) they may therefore be expreſſed by $2a + 1$, $2b + 1$ and $2c + 1$, reſpectively; $a$, $b$, and $c$ being Integers: In Conſequence of which Aſſumption, the Equation $4 n k^2 = p^2 - 4f$ will by Subſtitution be changed to $8 b c^2 + 8 b c + 2b + 4c^2 + 4c + 1 = 4a^2 + 4a + 1 - 4f$, or $2 b c^2 + 2 b c + \frac{1}{2} b + c^2 + c = a^2 + a - f$; from whence it is manifeſt, as all the Terms but $\frac{1}{2} b$ are known to be Integers, that $\frac{1}{2} b$ muſt be an Integer likewiſe: And ſo $b$ being an even Number, it follows that $n$, or $2 b + 1$, muſt be double of an even Number, (or a Multiple of 4) increaſed by Unity; therefore all the Diviſors of $\beta$ and $2\zeta$ that have not this Property, may be ſafely rejected, as not for the Purpoſe.

In like Manner, if $p$ be even the ſame Limitations will take Place, provided that $r$ is odd, which will be the Caſe when $\mathcal{Q}$ is the Half of an odd Number; for when $\mathcal{Q}$ is an Integer, $A^2 = \frac{1}{4} p^2 - f$, and $B^2 = \mathcal{Q}^2 - s$, being Integers, their Product $A^2 B^2$ will be an Integer, and conſequently, the ſquare Root thereof $A B$ being rational, will likewiſe be an Integer, and ſo $p \mathcal{Q}$ and $2 A B$ being both even Numbers, their Difference $r$, as

given by the Equation $p\,\mathcal{Q} = 2AB + r$, would be even, and not odd.

Let then $\mathcal{Q} = \frac{2x+1}{2}$, then $\mathcal{Q}^2 - s = \frac{4x^2+4x+1-4s}{4}$,

but $B^2$, or its Equal $n\,l^2$, being here equal to the Square of Half of an odd Number ($\mathcal{Q}$) joined to an Integer ($-s$), $l$ will be alſo Half of an odd Number.

Let then $l = \frac{2y+1}{2}$, then $n\,l^2 = \mathcal{Q}^2 - s = \frac{4ny^2+4ny+n}{4}$

conſequently $\frac{4x^2+4x+1-4s}{4} = n \times \left(\frac{4y^2+4y+1}{4}\right)$, and

$n = \frac{1+4x+4x^2-4s}{1+4y+4y^2} = 1 + 4 \times \left(\frac{x+x^2-s-y-y^2}{1+4y+4y^2}\right)$;

whence $n$ is poſitive, and the Double of an even Number, or a Multiple of 4 increaſed by Unity, when the Fraction $\frac{x+x^2-s-y-y^2}{1+4y+4y^2}$ is poſitive, but negative and a Multiple of 4 leſs Unity, when this Fraction is negative.

From thoſe ſeveral Concluſions is derived the following Rule for the Reduction of Equations of four Dimenſions, contained in the Formula

$$x^4 + p\,x^3 + q\,x^2 + r\,x + s = 0.$$

CII.

*Make* $\alpha = q - \frac{1}{8}p^2$, $\beta = r - \frac{1}{2}\alpha\,p$, *and* $\zeta = s - \frac{1}{4}\alpha\,\alpha$, *then put for* $n$ *ſome common integral Diviſor of* $\beta$ *and* $2\,\zeta$, *that is neither a Square, nor diviſible by a Square, and which being divided by* 4 *ſhall leave Unity, if* $n$ *be poſitive, but a Multiple of* 4 *leſs Unity if* $n$ *is negative, if either* $p$ *or* $r$ *be odd. Put alſo for* $k$ *ſome Diviſor of* $\frac{\beta}{n}$ *if* $p$ *be even, or Half of the odd Diviſor if* $p$ *be odd: Take the Quotient from* $\frac{1}{2}\,p\,k$, *and call Half the Remainder* $l$. *Make* $\mathcal{Q} = \frac{\alpha + n\,k^2}{2}$, *and try if* $n$ *divides* $\mathcal{Q}\,\mathcal{Q} - s$, *and the Root of the Quotient be equal to* $l$; *if ſo it happen, then the propoſed Equation, by Means of the Values thus determined, will be reduced to* $x\,x + \frac{1}{2}\,p\,x + \mathcal{Q} = \pm\sqrt{n} \times (k\,x + l)$.

General rule derived from the foregoing concluſions for reducing an equation of four dimenſions.

That the Diviſor $n$ ought not here to be a Square, is evident from what has been already remarked, ſince both $A$ and $B$ would then be rational Quantities; and that it ought not to be diviſible by a Square, will alſo appear, if it be conſidered that $k$ and $l$ in the Equation, $k\sqrt{n} = A$ and $l\sqrt{n} = B$ are to be taken the greateſt, and $n$ the leaſt, that the Caſe will admit of.

CIII.

Let the Equation $x^4 + 12\,x - 17 = 0$ be propoſe to be reduced.

Here $p = 0$, $q = 0$, $r = 12$, $s = -17$; consequently, $\alpha = 0$, $\beta = 12$, $\zeta = -17$, and $\beta$ and $2\zeta$; that is, 12 and $-34$, having only 2 for a common Divisor, it must be $n = 2$. Again $\frac{\beta}{n} = 6$, whose Divisors 1, 2, 3, 6, are to be successively put for $k$, and $-3$, $-\frac{3}{2}$, $-1$, $-\frac{1}{2}$, for $l$ respectively, but $\frac{\alpha + n k^2}{2}$; that is, $k^2$ is equal to $\mathfrak{Q}$, and $\sqrt{\frac{\mathfrak{Q}^2 - s}{n}} = l$; and when the even Divisors 2 and 6 are substituted for $k$, $\mathfrak{Q}$ becomes 4 and 36, and $\mathfrak{Q}^2 - s$ being an odd Number, is not divisible by $n = 2$; wherefore 2 and 6 are to be set aside: but when 1 and 3 are written for $k$, $\mathfrak{Q}$ is 1 or 9, and $\mathfrak{Q}^2 - s$ is 18 or 98 respectively; which Numbers can be divided by 2, and the Roots of the Quotients extracted being $\pm 3$ and $\pm 7$; but only one of them, Viz. $-3$ coincides with $l$. I put therefore $k = 1$, $l = -3$, $\mathfrak{Q} = 1$, and adding to both Sides of the Equation $n k^2 x^2 + 2 n k l x + n l^2$, that is, $2x^2 - 12x + 18$, there results $x^4 + 2x^2 + 1 = 2x^2 - 12x + 18$; and extracting the Root of each, $x^2 + 1 = \pm \sqrt{2} \times (x - 3)$, and again, extracting the Root of this last, the four Values of $x$ according to the Varieties in the Signs are $-\frac{1}{2}\sqrt{2} + \sqrt{(3\sqrt{2} - \frac{1}{2})}$, $-\frac{1}{2}\sqrt{2} - \sqrt{(3\sqrt{2} - \frac{1}{2})}$; $\frac{1}{2}\sqrt{2} + \sqrt{(-3\sqrt{2} - \frac{1}{2})}$, $\frac{1}{2}\sqrt{2} - \sqrt{(-3\sqrt{2} - \frac{1}{2})}$.

Application of the foregoing rule to an example.

CIV.

Let the Equation $x^4 - 6x^3 - 58x^2 - 114x - 11 = 0$ be proposed to be reduced.

Here $p = -6$, $q = -58$, $r = -114$, $s = -11$, consequently $-67 = \alpha$, $-315 = \beta$, and $-1135\frac{1}{4} = \zeta$. The Numbers $\beta$ and $2\zeta$, that is, $-315$ and $-\frac{4533}{2}$ have but one common Divisor 3, that is $n = 3$, and the Divisors of $-105 = \frac{\beta}{n}$ are 3, 5, 7, 15, 21, 35, and 105. Wherefore I first make Trial with $3 = k$ and dividing $\frac{\beta}{n}$ or $-105$ by it, get the Quotient $-35$, and this subtracted from $\frac{1}{2} p k = -3 \times 3$, leaves 26, whose half, 13, ought to be equal to $l$, but $\frac{\alpha + n k^2}{2}$ or $\frac{-67+27}{2}$ that is, $-20$ is equal to $\mathfrak{Q}$, and $\mathfrak{Q}^2 - s = 411$, which is indeed divisible by $n = 3$; but the Root of the Quotient 137 cannot be extracted, therefore I reject the Divisor 3, and try with $5 = k$, by which, dividing

The foregoing rule applied to another example.

$\frac{\beta}{n} = -105$ the Quotient is $-21$, and this taken from $\frac{1}{2} p k = -3 \times 5$, leaves $6 = 2l$. At the same Time $\mathcal{Q} = \frac{\alpha + n k^2}{2} = \frac{-67+75}{2} = 4$ and $\mathcal{Q}^2 - s$, or $16 + 11$ is divisible by $n$, and the Root of the Quotient 9, that is 3 coincides with $l$. Whence I conclude, that putting $l = 3$, $k = 5$, $\mathcal{Q} = 4$, $n = 3$, adding to both Sides of the Equation, the Quantity $n k^2 x^2 + 2 n k l x + n l^2$, that is, $75 x^2 + 90 x + 27$, and extracting the Roots, it will be

$x^2 + \frac{1}{2} p x + \mathcal{Q} = \sqrt{n} \times (k x + l)$, or $x^2 - 3x + 4 = \pm \sqrt{3} \times (5x + 3)$

and again extracting the Root, $x = \frac{3 \pm 5\sqrt{3}}{2} \pm \sqrt{17 \pm \frac{21 \times \sqrt{3}}{2}}$

CV.

No Regard in the foregoing Rule is had to that Circumstance in which $\beta$ happens to be nothing. In this Case, the Equation $\frac{1}{2} p k^2 - 2 k l = \frac{\beta}{n}$ becomes $\frac{1}{2} p k^2 - 2 k l = 0$; where one Root, or Value of $k$ must necessarily be nothing. Therefore $\mathcal{Q}$ being $= \frac{1}{2} \alpha$, we have $l \sqrt{n} = \sqrt{(\mathcal{Q}^2 - s)} = \sqrt{(\frac{1}{4} \alpha^2 - s)}$; so that by a direct Process our given Equation is reduced to $x^2 + \frac{1}{2} p x + \frac{1}{2} \alpha = \sqrt{(\frac{1}{4} \alpha \alpha - s)}$ wherein $\alpha$ is given $= q - \frac{1}{4} p p$.

Case in which the foregoing rule cannot be applied.

Rule to be observed in this case.

Thus the Equation $x^4 + 2 x^3 - 37 x^2 - 38 x + 1 = 0$, where $\alpha = -38$, $\beta = 0$, is reduced to $x^2 + x - 19 = + 6\sqrt{10}$.

Besides the Value of $k = 0$ when $\beta = 0$, there are two other Values which equally fulfil the several Conditions required, and bring out the very same Conclusion.

For substracting the Square of half the Second of the original Equations, $2\mathcal{Q} - \alpha = AA$, $p\mathcal{Q} - r = 2AB$, $\mathcal{Q}^2 - s = B^2$: from the Product of the other two, there will be obtained the Equation $\mathcal{Q}^3 + \frac{1}{2} q \mathcal{Q}^2 + (\frac{1}{4} p r - s) \times \mathcal{Q} + \frac{1}{2} \times (\alpha s - \frac{1}{4} r r) = 0$, wherein the unknown Quantity $\mathcal{Q}$ is alone concerned, which Equation being of three Dimensions, the Root $\mathcal{Q}$ and consequently $k = \frac{2\mathcal{Q} - \alpha}{n}$ will admit of three different Values.

Thus the Value of $k$ in the proposed Equation $x^4 + 2x^3 - 37x^2 - 38x + 1 = 0$ may be 0, 3, or 4, or which comes to the same, the Equation itself may be reduced to $x^2 + x - 19 = \pm 6\sqrt{10}$, $x^2 + x + \frac{7}{2} = \pm \sqrt{5} \times (3x + \frac{3}{2})$, or to $x^2 + x - 3 = \pm \sqrt{2} \times (4x + 2)$, all which are, in Effect, but one and the same Equation, as will appear by squaring both Sides of each,

Application of this rule to an example.

and properly transposing; from whence, in every Case, the given Equation $x^4 + 2x^3 - 37x^2 - 38x + 1 = 0$ will emerge.

CVI.

Third example.

Let the Equation $x^4 - 3x^3 + 44x^2 - 123x + 97 = 0$ be proposed to be reduced.

Here $p = -3$, $q = 44$, $r = -123$, $s = 97$, $q - \frac{3}{8}pp = 44 - \frac{9}{4} = \frac{167}{4}$, $\frac{pn}{2} = -\frac{501}{8}$, $r - \frac{1}{2}np = -123 + \frac{501}{8} = \frac{-984+501}{8} = -\frac{483}{8} = \beta$, $\frac{nn}{4} = +\frac{27889}{16}$; $s - \frac{1}{4}nn = 97 - \frac{27889}{4} = \frac{1552-27889}{16} = -\frac{26337}{16} = \xi$, the Divisors of $\beta$ are $-3$, $-7$, $-23$, and those of $\xi$ are $-3$, and $-8779$; whence $\beta$ and $2\xi$, having only $-3$ for a common Divisor, it must be $n = -3$; wherefore $\frac{\beta}{n} = +\frac{483}{8\times 3} = +\frac{161}{8}$: I therefore try whether $k = +\frac{7}{2}$ (because $k$ is an odd Number) will succeed. $\frac{pk}{2} - \frac{\beta}{nk} = -\frac{21+23}{4} = -\frac{44}{4} = -11$; and $l = -\frac{11}{2}$. Whence is deduced $\frac{\alpha + nk^2}{2} = \frac{147}{8} - \frac{3}{2}\times\frac{49}{4} = \frac{167-147}{8} = +\frac{5}{2} = \mathfrak{Q}$, and $\mathfrak{Q}^2 = \frac{25}{5}$; $\mathfrak{Q}^2 - s = \frac{25-388}{8} = -\frac{363}{4}$. This Number divided by $n = -3$ gives $\frac{121}{4}$, whose Root is $\pm\frac{11}{2} = l$; wherefore the proposed Equation can be reduced, and there results

$$x^2 - \tfrac{3}{2}x + \tfrac{5}{2} = \left(\tfrac{7}{2}x - \tfrac{11}{2}\right)\sqrt{-3}.$$

CVII.

Application of the foregoing rule for reducing equations having fracted terms.

If the highest Term of the Equation to be reduced has a Coeficient different from Unity, the Equation may be transformed into one that shall have the Coeficient of the highest Term Unity, and the foregoing Rules may be applied to the new Equation.

Thus, if the Equation $mx^4 + px^3 + qx^2 + rx + s = 0$ be multiplied by $m^3$, it will be transformed into

$$m^4x^3 + m^3px^3 + m^3qx^2 + m^3rx + m^3s = 0.$$

Let $m x = y$, and $y^4 + p y^3 + m q y y + m m r y + m^3 s = 0$ will result; to which the foregoing Rules may be applied.

For Example, if the Equation $2x^4 - 6x^3 + 137x^2 - 400x + 315 = 0$ was propoſed, here $2x = y$, and conſequently $m = 2$, $m^2 = 4$; $m^3 = 8$. Subſtituting $y$ for $x$, and multiplying the Coeficient of the third Term by 2, the fourth by 4, and the laſt by 8, there will reſult,

$$y^4 - 6y^3 + 274y^2 - 1600y + 2520 = 0.$$

Here $p = -6$, $q = 274$, $r = 1600$, $s = 2520$, $q - \frac{1}{4}p^2 = 274 - 9 = 265 = a$, $\frac{pa}{2} = -3 \times 265 = -795$, and $r - \frac{1}{2}pa =$ Example.
$-1600 + 795 = -805 = \beta$, $s - \frac{1}{4}aa = 2520 - \frac{70225}{4} = -\frac{60145}{4} = \zeta$.

The Diviſors of $\beta$ are $-5$, $-7$, $-23$, and thoſe of $\zeta$ are $-5$, $-23$, $-523$; wherefore $n = -5$, or $n = -23$. I put $n = -5$, whence $\frac{\beta}{n} = \frac{-805}{-5} = +161$, and $k = 7$, or $k = 23$.

Let $k = 7$, $\frac{pk}{2} - \frac{\beta}{nk} = -21 - 23 = -44 = 2l$, and $l = -22$; whence is deduced $\frac{a + nk^2}{2} = \frac{265 - 5 \times 49}{2} = 10 = \mathfrak{Q}$, and $\mathfrak{Q}\mathfrak{Q} = 100$, $\mathfrak{Q}\mathfrak{Q} - s = 100 - 2520 = -2420$, and $\frac{\mathfrak{Q}\mathfrak{Q} - s}{n} = \frac{-2420}{-5} = 484 = (-22)^2 = l^2$; the Equation therefore reduced is $y^2 - 3y + 10 = (7y - 22)\sqrt{-5}$, and reſtoring $4x^2$ for $y^2$ and $2x$ for $y$, and dividing all the Terms by 4, $x^2 - \frac{3}{2}x + \frac{5}{2} = \left(\frac{7}{2}x - \frac{11}{2}\right)\sqrt{-5}$.

CVIII.

Application of the foregoing rule for extracting the rational roots of equations of four dimenſions.

Hitherto we have applied the Rule to the Extraction of ſurd Roots, to apply the ſame to the Extraction of rational Roots, it ſuffices to make uſe of Unity for the Quantity $n$, and by this Means we may find whether an Equation that has no fracted or ſurd Terms can admit of any Diviſor, either rational or ſurd, of two Dimenſions.

Thus, if the Equation $x^4 - x^3 - 5x^2 + 12x - 6 = 0$ was propoſed, by ſubſtituting $-1$, $-5$, $+12$, and $-6$, for $p$, $q$, $r$ and $s$ re-

ſpectively, I find $\alpha = -5\frac{1}{4}$, $\beta = 9\frac{3}{8}$, and putting $n = 1$, the Diviſors of the Quantity $\frac{\beta}{n}$, or $\frac{75}{8}$, are 1, 3, 5, 15, 25, 75, the Halves whereof (if $p$ be odd) are to be tried for $k$, and for $k$ trying $\frac{5}{2}$, I find $\frac{1}{2}pk - \frac{\beta}{nk} = -5$, and its Half $-\frac{5}{8} = l$. Alſo $\frac{\alpha + nkk}{2} = \frac{1}{8} = \mathcal{Q}$, and $\frac{\mathcal{Q}\mathcal{Q} - s}{n} = 6\frac{1}{4}$, the Root whereof agrees with $l$. I therefore conclude, that the Quantities $n$, $k$, $l$, $\mathcal{Q}$, are rightly found, and having added to each Part of the Equation the Terms $nk^2x^2 + 2nklx + nll$, that is, $6\frac{1}{4}xx - 12\frac{1}{2}x + 6\frac{1}{4}$, the Root may be extracted on both Sides, and by that Extraction there will come out $xx + \frac{1}{2}px + \mathcal{Q} = \pm\sqrt{n} \times (kx + l)$, that is, $xx - \frac{1}{2}x + \frac{1}{2} = \pm 1 \times (2\frac{1}{2}x - 2\frac{1}{2})$, or $xx - 3x + 3 = 0$, and $xx + 2x - 2 = 0$, and ſo by theſe two quadratick Equations the biquadratick One propoſed may be divided.

CIX.

If at any Time there are many Diviſors of the Quantity $\frac{\beta}{n}$, ſo that it may be too difficult to try all of them for $k$, their Number may be diminiſhed by ſeeking all the Diviſors of the Quantity $\alpha s - \frac{1}{4}rr$. For the Quantity $\mathcal{Q}$ ought to be equal to ſome of thoſe, or to the Half of ſome odd one; for $\alpha s = \alpha\mathcal{Q}^2 - \alpha l n^2$, and taking from both $\frac{1}{4}r^2 = \frac{1}{4}p^2\mathcal{Q}^2 - p\mathcal{Q}nkl + n^2k^2l^2$, ſeeing the Remainder $\alpha\mathcal{Q}^2 - (\alpha + nk^2) \times nl^2 - \frac{1}{4}p^2\mathcal{Q}^2 + p\mathcal{Q}nkl = \alpha\mathcal{Q}^2 - 2\mathcal{Q} \times nl^2 - \frac{1}{4}p^2\mathcal{Q}^2 + p\mathcal{Q}nkl$ has $\mathcal{Q}$ in every Term, the Thing is manifeſt.

Obſervation which ſerve to diminiſh the number of trials.

Thus in the laſt Example $\alpha s - \frac{1}{4}rr$ is $-\frac{9}{2}$, ſome one of whoſe Diviſors 1, 3, 9, or of them halved $\frac{1}{2}$, $\frac{3}{2}$, $\frac{9}{2}$, ought to be $\mathcal{Q}$; Wherefore by trying ſingly the halved Diviſors of the Quantity $\frac{\beta}{n}$ Viz. $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, $\frac{15}{2}$, $\frac{25}{2}$ and $\frac{75}{2}$ for $k$. I reject all that do not make $\frac{1}{2}\alpha + \frac{1}{2}nk^2$, or $\frac{21}{8} + \frac{1}{2}kk$; that is, $\mathcal{Q}$ to be one of

the Numbers 1, 3, 9, $\frac{1}{2}$, $\frac{3}{2}$, $\frac{9}{2}$; but by writing $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, $\frac{15}{2}$, &c. for $k$, there come out reſpectively $-\frac{5}{2}$, $-\frac{3}{2}+\frac{1}{2}+\frac{51}{2}$, &c. for $\mathfrak{Q}$, out of which only $-\frac{3}{2}$ and $\frac{1}{2}$ are found among the aforeſaid Numbers 1, 3, 9, $\frac{1}{2}$, $\frac{3}{2}$, $\frac{9}{2}$, and conſequently the reſt being rejected, either $k$ will be $=\frac{3}{2}$, and $\mathfrak{Q}=-\frac{3}{2}$, or $k=\frac{5}{2}$ and $\mathfrak{Q}=\frac{1}{2}$; which two Caſes let be examined.

CX.

If the Equation to be reduced is of ſix Dimenſions, let it be $x^6+px^5+qx^4+rx^3+sx^2+tx+v=0$, and let there be aſſumed $(x^3+\frac{1}{2}px^2+\mathfrak{Q}x+R)^2-(Ax^2+Bx+C)^2=x^6+px^5+qx^4+rx^3+sx^2+tx+v=0$, which, by Involution and Tranſpoſition, will give

$(2\mathfrak{Q}+\frac{1}{4}p^2-q)x^4+(2R+p\mathfrak{Q}-r)x^3+(pR+\mathfrak{Q}^2-s)x^2+(2\mathfrak{Q}R-t)x+R^2-v=A^2x^4+2ABx^3+(2AC+B^2)x^2+2BCx+C^2$.

Inveſtigation of the method of reducing equations of ſix dimenſions by trial.

From whence by equating the Coeficients of the homologous Powers, and writing $\alpha=q-\frac{3}{4}pp$,

We have 1°. $2\mathfrak{Q}-\alpha=A^2$; 2°. $2R+p\mathfrak{Q}-r=2AB$; 3°. $pR+\mathfrak{Q}^2-s=2AC+B^2$; 4°. $2\mathfrak{Q}R-t=2BC$; 5°. $R^2-v=C^2$.

If now the Value of $\mathfrak{Q}=\frac{1}{2}A^2+\frac{1}{2}\alpha$, as given by the firſt of theſe Equations, be ſubſtitued for $\mathfrak{Q}$, in the ſecond, we ſhall get $2R+\frac{1}{2}pA^2+\frac{1}{2}p\alpha-r=2AB$, and conſequently, $R=AB-\frac{1}{4}pA^2+\frac{1}{2}\beta$ (by putting $\beta=r-\frac{1}{2}p\alpha$), which Value, together with that of $\mathfrak{Q}$, being ſubſtituted in the three remaining Equations, we ſhall have,

1°. $pAB-\frac{1}{4}p^2A^2+\frac{1}{2}p\beta+\frac{1}{4}A^4+\frac{1}{2}\alpha A^2+\frac{1}{4}\alpha^2-s=2AC+B^2$

2°. $A^3B-\frac{3}{4}pA^4+\frac{1}{2}\beta A^2+\alpha AB-\frac{1}{2}p\alpha A^2+\frac{1}{2}\alpha\beta-t=2BC$

$3°.\ A^2B^2 - \frac{1}{2}pA^3B + \beta AB + \frac{1}{16}p^2A^4 - \frac{1}{4}p\beta A^2 + \frac{1}{4}\beta^2 - v = C^2$

which by putting $\gamma = \varepsilon - \frac{1}{2}p\beta$, $\zeta = \gamma - \frac{1}{4}\alpha\alpha$, $\eta = \iota - \frac{1}{2}\alpha\beta$, and $\theta = v - \frac{1}{4}\beta\beta$, will be reduced to

$$pAB - \tfrac{1}{4}p^2A^2 + \tfrac{1}{4}A^4 + \tfrac{1}{2}\alpha A^2 - \zeta = 2AC + B^2$$

$$A^3B - \tfrac{1}{2}pA^4 + \tfrac{1}{2}\beta A^2 + \alpha AB - \tfrac{1}{2}p\alpha A^2 - \eta = 2BC$$

and $A^2B^2 - \frac{1}{2}pA^3B + \beta AB + \frac{1}{10}p^2A^4 - \frac{1}{2}p\beta A^2 - \theta = C^2$

respectively.

Now as the Values of $A$, $B$, and $C$, are either rational Quantities, or if irrational, affected by the same radical Quantity; let them be expressed by $k\sqrt{n}$, $l\sqrt{n}$, and $m\sqrt{n}$, respectively; then Substitution being made, and every Equation divided by $n$, we shall have

$$1°.\ pkl - \frac{1}{4}p^2k^2 + \frac{1}{4}nk^4 + \frac{1}{2}\alpha k^2 - \frac{\zeta}{n} = 2km + l^2$$

$$2°.\ nk^3l - \frac{1}{2}pnk^4 + \frac{1}{2}\beta k^2 + \alpha kl - \frac{1}{4}p\alpha k^2 - \frac{\eta}{n} = 2lm$$

$$3°.\ nk^2l^2 - \frac{1}{2}pnk^3l + \beta kl + \frac{1}{16}p^2nk^4 - \frac{1}{4}p\beta k^2 - \frac{\theta}{n} = m^2$$

From whence it appears (since $k$, $l$, and $m$, are here considered as Integers, or as the Halves of such) that $\zeta$, $\eta$, and $\theta$, ought to be all of them divisible by $n$, or which is the same, that $n$ ought to be some common integral Divisor of the Quantities $\zeta$, $\eta$, and $\theta$.

With Regard to $k$, let the several Terms in the former Part of the first of our three Equations, in which $k$ is found, (in order to abreviate the Work) be denoted by $Fk$, then will the Equation itself be changed to $Fk - \frac{\zeta}{n} = 2km + l^2$, and in the very same Manner our other two Equations will be changed to $Gk - \frac{\eta}{n} = 2lm$, and $Hk - \frac{\theta}{n} = m^2$, respectively.

Let now the Square of Half the Second of these Equations be subtracted from the Product of the first and third, then will

$$FHk^2 - \frac{Fk\theta}{n} - \frac{Hk\zeta}{n} + \frac{\zeta\theta}{nn} - \frac{1}{4}G^2k^2 + \frac{Gk\eta}{2n} - \frac{\frac{1}{4}\eta^2}{nn} = 2km^2,$$

which, by dividing the whole by $2k$, and putting $\lambda = \zeta\theta - \frac{1}{4}\eta^2$, will at Length become,

$$\frac{1}{2}FHk - \frac{1}{8}G^2k - \frac{1}{2}H \times \frac{\zeta}{n} + \frac{1}{4}G \times \frac{\eta}{n} - \frac{1}{2}F \times \frac{\theta}{n} + \frac{\frac{\lambda}{2nn}}{k} = m^2;$$

where $\zeta$, $\eta$, and $\theta$, being all divisible by their common Divisor $n$, (as is shewn above); it is manifest, that $\frac{\lambda}{2nn}$ (in order that $m$ may be an Integer, or the Half of an Integer), ought also to be divisible by its Divisor $k$; that is, $k$ ought to be some Divisor of the Quantity $\frac{\lambda}{2nn}$.

Again, with Regard to $l$, let the Value of $R = AB - \frac{1}{2}pQ + \frac{1}{2}r$, as given by the second of the five original Equations, be substituted in the fourth, by which Means we have $2QAB = pQ^2 + rQ - t = 2BC$, or $2Qnkl - pQ^2 + rQ - t = 2nlm$ (because $A = k\sqrt{n}$, $B = l\sqrt{n}$, $C = m\sqrt{n}$) and consequently $\frac{rQ - pQ^2 - t}{nl} = 2m - 2kQ$; where $2m - 2kQ$, being an Integer, it is evident that $\frac{rQ - pQ^2 - t}{nl}$ must be an Integer also; and consequently, $l$ some Divisor of the Quantity $\frac{rQ - pQ^2 - t}{n}$, from whence $l$ being found in Numbers, the Value of $R = AB - \frac{1}{2}pQ + \frac{1}{2}r = nkl - \frac{1}{2}pQ + \frac{1}{2}r$ will be had likewise; and then by Means of the three last of the five original Equations, the Value of $m$ may be also found three different Ways, and the Truth of the Solution thereby confirmed: For those Equations, by substituting for $A$, $B$, and $C$, their Equals $k\sqrt{n}$, $l\sqrt{n}$, and $m\sqrt{n}$, do become $R^2 - v = nm^2$, $2QR - t = 2nlm$, and $pR + Q^2 - s = 2nkm + nll$; from the first of which $m = \sqrt{\frac{R^2 - v}{n}}$; from the second $m = \frac{QR - \frac{1}{2}t}{nl}$, and from the third $m = \frac{2Q + pR - nll - s}{2nk}$, which Values therefore, when $Q$, $R$, &c. are rightly assumed, will be all found equal among themselves.

CXI.

As the foregoing Method of Solution depends upon the assuming proper Divisors of $\zeta$, $\eta$, $\theta$, &c. for $n$, &c. To bring the Work into less Compass, it will be expedient to reject such Divisors of those Quantities as are not to the Purpose, and this may be effected as in the precedent Case, from the Consideration of the Properties of even and odd Numbers, the Reasoning thereon being the very same in this Case as in the former, which therefore it will be unnecessary to repeat. And from the Conclusions above derived is deduced the following Rule for reducing an Equation of six Dimensions $x^6 + px^5 + qx^4 + rx^3 + sx^2 + tx + o = 3$.

The number of trials diminished from the consideration of the properties of even and odd numbers.

CXII.

General rule derived from the foregoing method for reducing an equation of six dimensions.

*Make* $q - \frac{5}{4} p p = \alpha$, $r - \frac{1}{2} p \alpha = \varepsilon$, $s - \frac{1}{2} p \beta = \gamma$, $\gamma - \frac{5}{4} \alpha \alpha = \zeta$, $t - \frac{1}{2} \alpha \beta = \eta$, $v - \frac{1}{4} \beta \beta = \theta$, $\zeta \theta - \frac{1}{4} \eta \eta = \lambda$.

*Then for* $n$ *take of the Terms* $2\zeta$, $\eta$, $2\theta$, *ſome common Integer Diviſor, that is not a Square, and that likewiſe is not diviſible by a Square, and which alſo divided by the Number* 4, *ſhall leave Unity, provided any one of the Terms* $p$, $r$, $t$, *be odd. For* $k$ *take ſome Integer Diviſor of the Quantity* $\frac{\lambda}{2nn}$ *if* $p$ *be even, or the Half of an odd Diviſor, if* $p$ *be odd, for* $Q$ *take the Quantity,* $\frac{1}{2}\alpha + \frac{1}{2} n k k$, *take for* $l$ *ſome One of the Diviſors of the Quantity* $\frac{2r - Qq - t}{n}$, *if* $Q$ *be an Integer, or the Half of an odd Diviſor, if* $Q$ *be a Fraction that has for its Denominator the Number* 2; *or* 0, *if that Dividual* $\frac{2r - Qq - t}{n}$ *be nothing. And for* $R$ *the Quantity* $\frac{1}{2} r - \frac{1}{2} Q p + n k l$. *Then try if* $RR - v$ *can be divided by* $n$, *and the Root of the Quotient extracted; and beſides, if that Root be equal as well to the Quantity* $\frac{2R - \frac{1}{2} t}{n l}$ *as to the Quantity* $\frac{2Q + pR - nll - s}{2nk}$; *if all theſe happen, call that Root* $m$, *and in Room of the Equation propoſed, write this* $x^3 + \frac{1}{2} p x x + Q x + R = \pm \sqrt{n} \times (k x^2 + l x + m)$; *for this Equation, by ſquaring its Parts, and taking from both Sides the Terms on the right Hand, will produce the Equation propoſed.*

CXIII.

Application of the foregoing rule to an example.

Let $x^6 - 2 a x^5 + 2 b b x^4 + 2 a b b x^3 - (2 a a b b - 2 a^3 b + 4 a b^3) x^2 + 3 a a b^4 - a^4 b b = 0$ be propoſed to be reduced.

Here writing $-2a$, $+2bb$, $+2abb$, $-2aabb + 2a^3b - 4ab^3$, 0, and $3aab^4 - a^4bb$ for $p$, $q$, $r$, $s$, $t$, and $v$, reſpectively, there will come out $2bb - aa = \alpha$, $4abb - a^3 = \varepsilon$, $2a^3b + 2aabb - 4ab^3 - a^4 = \gamma$, $-b^4 + 2a^3b + 3aabb - 4ab^3 - \frac{5}{4} a^4 = \zeta$. $-\frac{1}{2} a^5 - 3a^3bb - 4ab^4 = \eta$, and $-aab^4 + a^4bb - \frac{1}{4} a^5 = \theta$; and the common Diviſor of the Terms $2\zeta$, $\eta$ and $2\theta$, is $aa - 2bb$, or $2bb - aa$, according as $aa$, or $2bb$, is the greater. But let $aa$ be greater than $2bb$, and $aa - 2bb$ will be $n$; for $n$ muſt always be affirmative. Moreover we have, $\frac{\zeta}{n} = -\frac{5}{4} aa + 2ab + \frac{1}{2} bb$, $\frac{\eta}{n} = -\frac{1}{2} a^3 + 2abb$, and $\frac{\theta}{n} = -\frac{1}{4} a^4 + \frac{1}{2} aabb$, and

$\frac{\zeta}{2n} \times \frac{\theta}{n} - \frac{\eta\eta}{8nn}$, or $\frac{\lambda}{2nn} = \frac{1}{8} a^6 - \frac{1}{4} a^5 b - \frac{1}{8} a^4 bb + \frac{1}{2} a^3 b^3 - \frac{3}{8} a^2 b^4$, the Diviſors whereof are 1, $a$, $aa$; but becauſe $\sqrt{n} \times k$ cannot be of more than one Dimenſion, and that $\sqrt{n}$ is of one, therefore $k$ will be of none; and, conſequently, can only be a Number. Wherefore, rejecting $a$ and $aa$, there remains only 1 for $k$, beſides $\frac{1}{2} \alpha + \frac{1}{2} n k k$ gives 0 for $Q$, and $\frac{2r - 2Qp - s}{n}$ is alſo Nothing; and conſequently $l$, which ought to be its Diviſor, will be Nothing. Laſtly, $\frac{1}{2} r - \frac{1}{2} p Q + n k l$ gives $abb$ for $R$, and $RR - v$ is $-2aab^4 + a^4 bb$, which may be divided by $n$, or $aa - 2bb$, and the Root of the Quotient $aabb$ be extracted; and that Root taken negatively, Viz. $-ab$ is not unequal to the indefinite Quantity $\frac{2R - \frac{1}{2} p s}{nl}$, or $\frac{0}{0}$, but equal to the definite Quantity $\frac{2Q + pR - nll - s}{2nk}$, wherefore that Root $-ab$ will be $m$, and in the Room of the Equation propoſed, there may be writ $x^3 - \frac{1}{2} p x x + Qx + R = \sqrt{n} \times (k xx + lx + m)$, that is, $x^3 - axx + abb = \sqrt{(aa - 2bb)} \times (xx - ab)$; the Truth of which Concluſion you may prove by ſquaring the Parts of the Equation found, and taking away the Terms on the right Hand from both Sides; for from that Operation will be produced the Equation $x^6 - 2axx^5 + 2bbx^4 + 2abbx^3 - (2aabb - 2a^3 b + 4ab^3) x^2 + 3aab^4 - a^4 bb = 0$, which was propoſed to be reduced.

CXIV.

When $\lambda = 0$; in which Caſe $k$ (or one Value of $k$ at leaſt) will alſo be $= 0$, the Reduction will be performed by a direct Proceſs. For $k$ being Nothing, the three Equations wherein $k$, $l$, and $m$ are firſt introduced will become $-\frac{\zeta}{n} = l^2$, $-\frac{\eta}{n} = 2lm$, and $-\frac{\theta}{n} = m^2$; whence $l\sqrt{n} = \sqrt{-\zeta}$, $m\sqrt{n} = \sqrt{-\theta}$; and conſequently $\lambda = \zeta\theta - \frac{1}{4}\eta^2 = 0$, as it ought to be; therefore, by ſubſtituting thoſe Values, and writing alſo, inſtead of $Q$ and $R$, their Equals $\frac{1}{2}\alpha$, and $\frac{1}{2}\beta$, the Equation given is here reduced to $x^3 + \frac{1}{2} p x^2 + \frac{1}{2} \alpha x + \frac{1}{4} \beta = \pm x \sqrt{-\zeta} \pm \sqrt{-\theta}$.

Caſe in which the foregoing rule cannot be applied. Rule to be obſerved in this caſe.

CXV.

If the Equation propoſed is of eight Dimenſions, let it be $x^8 + p x^7 + q x^6 + r x^5 + s x^4 + t x^3 + v x^2 + w x + z = 0$; then by aſſuming $(x^4 + \frac{1}{2} p x^3 + Q x^2 + Rx + S)^2 - (Ax^3 + Bx^2 + Cx + D)^2 = x^8 + p x^7 + q x^6 + r x^5 + s x^4 + t x^3 + v x^2 + w x + z = 0$;

and proceeding as in the former Cases we shall have here, 1°. $2Q - \alpha = A^2$, 2°. $2R + pQ - r = 2AB$, 3°. $2S + pR + QQ - s = 2AC + BB$, 4°. $pS + 2QR - t = 2AD + 2BC$, 5°. $2QS + RR - v = 2BD + CC$, 6°. $2RS - w = 2CD$. 7°. $SS - z = DD$.

Method for discovering whether an equation of eight dimensions can be reduced.

Putting now (as before) $A = k\sqrt{n}$, $B = l\sqrt{n}$, $C = m\sqrt{n}$, $D = b\sqrt{n}$; putting also, to shorten the Work, $Q = Q'n + \frac{1}{2}\alpha$ $R = R'n + \frac{1}{2}\beta$, $S = S'n + \frac{1}{2}\gamma$; that is, let the Quotients of $Q$, $R$, and $S$, when divided by the common Divisor $n$, be $Q'$, $R'$, $S'$, and the Remainders $\frac{1}{2}\alpha$, $\frac{1}{2}\beta$, and $\frac{1}{2}\gamma$, respectively: Then to determine these last, which must be first known, before $n$ can be known, let Substitution be made in the second and third Equations, every where disregarding such Terms wherein $n$ and its Powers are involved. Thus Substitution being made in the second Equation, we have $2R'n + \beta + pQ'n + \frac{1}{2}p\alpha - r = 2kln$; where the homologous Terms in which $n$ enters not are $\beta$, $\frac{1}{2}p\alpha$, and $-r$: The others therefore being here disregarded, we have $\beta + \frac{1}{2}p\alpha - r = 0$, or $\beta = r - \frac{1}{2}p\alpha$; in the very same Manner from the third Equation $\gamma + \frac{1}{2}p\beta + \frac{1}{4}\alpha\alpha - s = 0$; and consequently, $\gamma = s - \frac{1}{2}p\beta - \frac{1}{4}\alpha\alpha$.

Let Substitution be now made in the fourth, fifth, sixth, and seventh, Equations (still disregarding such Terms as would involve the Powers of $n$) and there will come out, 1°. $\frac{1}{2}p\gamma + \frac{1}{2}\alpha\beta - t$, 2°. $\frac{1}{2}\alpha\gamma + \frac{1}{4}\beta\beta - v$, 3°. $\frac{1}{2}\beta\gamma - w$, 4°. $\frac{1}{4}\gamma\gamma - z$. Now, as all the other Terms that would arise in these Equations (besides those put down) are affected with $n$, and are therefore divisible thereby; it is manifest, that the four Quantities $t - \frac{1}{2}p\gamma - \frac{1}{2}\alpha\beta$, $v - \frac{1}{2}\alpha\gamma - \frac{1}{4}\beta\beta$, $w - \frac{1}{2}\beta\gamma$, and $z - \frac{1}{4}\gamma\gamma$; here brought out, which, for Brevity sake, I represent severally by $\delta$, $\varepsilon$, $\zeta$, $\eta$, must likewise be all of them divisible, by the same common Divisor $n$, when the Equation given is capable of being reduced. If therefore no such common Divisor, (under the Restrictions specified in the preceding Cases, depending on any Coefficient $p$, $r$, $t$, or $w$, in a Place denominated from an even Number, being an odd Number,) can be discovered, the Work will then be at an End.

## CXVI.

From the same Method of Operation, which may be looked upon as a sort of Examination, whether the Equation be reducible or not, we may find all the Quantities, to which $n$ ought to be a common Divisor, when the Equation given is of 10, 12, or a greater Number of Dimensions.

General method for discovering whether an equation of

Let $x^{2m} + px^{2m-1} + qx^{2m-2} + rx^{2m-3} + sx^{2m-4} + tx^{2m-5} + \&c. = 0$ be given, and let there be assumed,

$[x^m + \frac{1}{2}px^{m-1} + (Q'n + \frac{1}{2}\alpha)x^{m-2} + (R'n + \frac{1}{2}\beta)x^{m-3} + (S'n + \frac{1}{2}\gamma)x^{m-4}$ &c.$]^2$ any number of even dimensions can be reduced.
$- n(kx^{m-1} + lx^{m-2} + mx^{m-3}$ &c.$)^2 = x^{2m} + px^{2m-1} + qx^{2m-2}$ &c. then by squaring $x^m + \frac{1}{2}px^{m-1} + (Q'n + \frac{1}{2}\alpha)x^{m-2}$, &c. and transposing $x^{2m} + px^{2m-1} + qx^{2m-2}$, &c. it will appear that the Terms of this Equation, in which $n$ enters not, will be,

$$\left.\begin{matrix}\alpha \\ \frac{1}{4}p^2 \\ -q\end{matrix}\right\}\times x^{2m-2} + \left.\begin{matrix}\beta \\ \frac{1}{2}p\alpha \\ -r\end{matrix}\right\}\times x^{2m-3} + \left.\begin{matrix}\gamma \\ \frac{1}{2}p\beta \\ \frac{1}{4}\alpha^2 \\ -s\end{matrix}\right\}\times x^{2m-4} + \left.\begin{matrix}\delta \\ \frac{1}{2}p\gamma \\ \frac{1}{2}\alpha\beta \\ -t\end{matrix}\right\}\times x^{2m-5} + \left.\begin{matrix}\varepsilon \\ \frac{1}{2}p\delta \\ \frac{1}{2}\alpha\gamma \\ \frac{1}{4}\beta\beta \\ -v\end{matrix}\right\}\times x^{2m-6}$$

From the former Half of which Terms all the Quantities $\alpha$, $\beta$, $\gamma$, &c. will be determined by assuming the Coeficients equal to Nothing: Thus we have $\alpha = q - \frac{1}{4}pp$, $\beta = r - \frac{1}{2}p\alpha$, $\gamma = s - \frac{1}{2}p\beta - \frac{1}{4}\alpha\alpha$, $\delta = t - \frac{1}{2}p\gamma - \frac{1}{2}\alpha\beta$, &c. and those Quantities being known, the Coeficients of the remaining Terms will likewise be known, which ought all of them to be divisible by $n$, in Order that the Reduction may succeed; that is, they ought to be such, as to admit of a common Divisor $(n)$ under the Restrictions before specified.

CXVII.

For Example, if the Equation given is of twelve Dimensions, as $x^{12} + px^{11} + qx^{10} + rx^9 + sx^8 + tx^7 + vx^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$, we should have $\alpha = q - \frac{1}{4}pp$ Application of the foregoing method to an example.
$\beta = r - \frac{1}{2}p\alpha$, $\gamma = s - \frac{1}{2}p\beta - \frac{1}{4}\alpha\alpha$, $\delta = t - \frac{1}{2}p\gamma - \frac{1}{2}\alpha\beta$, and $\varepsilon = v - \frac{1}{2}p\delta - \frac{1}{2}\alpha\gamma - \frac{1}{4}\beta\beta$, and the Coeficients of the other six Terms (whereof $n$ ought to be a common Divisor) would be
$\frac{1}{2}p\varepsilon + \frac{1}{2}\alpha\delta + \frac{1}{2}\beta\gamma - a$, $\frac{1}{2}\alpha\varepsilon + \frac{1}{2}\beta\delta + \frac{1}{4}\gamma\gamma - b$, $\frac{1}{2}\beta\varepsilon + \frac{1}{2}\gamma\delta - c$, $\frac{1}{2}\gamma\varepsilon + \frac{1}{4}\delta\delta - d$, $\frac{1}{2}\delta\varepsilon - e$, and $\frac{1}{4}\varepsilon\varepsilon - f$.

These Operations for finding of $n$ will most commonly end the Work. If such a Value however should be found for $n$ as to answer all the Conditions above specified, and consequently there are Hopes of a future Reduction. It may be tried by the following Steps, which though applied only for reducing an Equation of eight Dimensions, yet it is easy to perceive, are general for all Equations.

CXVIII.

Because by the 7th Equation $SS - z = nb^2$, if $n$ be even, seek a square Number $b^2$, to which, after it is multiplied into $n$, the last Term of the Equation $z$, being added with its proper Sign $z + nb^2$, shall make

a ſquare Number, which may be expeditiouſly performed, by adding to $z$ when $n$ is an even Number, or to $4z$ when it is odd, thoſe Quantities ſucceſſively $n$, $3n$, $5n$, $7n$, $9n$, $11n$, and ſo on till the Sum becomes equal to ſome Number in the Table of ſquare Numbers; and if no ſuch Number occurs, before the ſquare Root of that Sum, augmented by the ſquare Root of the Exceſs of that Sum, above the laſt Term of the Equation, is four Times greater than the greateſt of the Terms of the propoſed Equation $p$, $q$, $r$, $s$, $t$, $v$, &c. There will be no Occaſion to try any farther; for then the Equation cannot be reduced: But if ſuch a ſquare Number does occur, let its Root be $S$, if $n$ be even, or $2S$, if $n$ be odd; and call $\sqrt{\frac{SS-z}{n}} = b$; but if $n$ is odd, then $S$ and $b$ may be Fractions that have 2 for their Denominator: And if one is a Fraction, the other ought to be ſo too, which is alſo to be obſerved of the Numbers $R$ and $m$, $Q$ and $l$, $p$ and $k$, hereafter to be found. And all the Numbers $S$ and $b$ within this Limit muſt be collected in a Catalogue.

General rule for reducing equations that have been found by the foregoing method to be reducible

Having thus found $n$, $b$, and $S$; $k$ is next to be found by ſucceſſively trying all Numbers which do not make $nk \pm \frac{1}{2}p$ four Times greater than the greateſt Term of the Equation; when $k$ is had, $Q$ is to be found by the Equation $Q = \frac{nk^2 + \alpha}{2}$, $Q$ being found, all Numbers are to be tried for $l$ which do not make $nl \pm Q$ greater than quadruple the greateſt Term of the Equation, and $l$ being found we have $R = \frac{-npk^2 + 2\beta}{4} + nkl$. Laſtly to find $m$, all Numbers are ſucceſſively to be tried, which do not make $nm \pm R$ greater than quadruple the greateſt Term of the Equation. Of the Values of the Letters regiſtered in the Catalogue, thoſe only are to be aſſumed which agree when found ſeveral Ways, thus, $S = \sqrt{z + nb^2}$ by the 7th Equation; and alſo $= \frac{s - pR - QQ + nll}{2} + nkm$ by the third; and its Correſpondent $b = \frac{\sqrt{SS-z}}{n}$ by the 7th Equation, and $= \frac{pS + 2QR - t - 2nlm}{2nk}$ by the fourth; alſo $= \frac{2QS + RR - v - nmm}{2nl}$ by the 5th, and $= \frac{2RS - w}{2nm}$ by the 6th Equation; which Values therefore, when $Q$, $R$, &c. are rightly aſſumed, will be all found equal among themſelves, and the Equation propoſed will be then reduced to

$$x^4 + \tfrac{1}{2}px^3 + Qx^2 + Rx + S = \sqrt{n} \times (kx^3 + lx^2 + mx + b)$$

CXIX.

Let $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10xx - 10x - 5 = 0$, be propoſed to be reduced.

Here $q - \frac{1}{4}pp = -1 - 4 = -5 = \alpha$, $r - \frac{1}{2}p\alpha = -10 + 10 = 0 = \beta$, $s - \frac{1}{2}p\beta - \frac{1}{4}\alpha\alpha = 5 - \frac{25}{4} = -\frac{5}{4} = \gamma$, $t - \frac{1}{2}p\gamma - \frac{1}{2}\alpha\beta = -5 + \frac{5}{2} = -\frac{5}{2} = \delta$, $v - \frac{1}{2}\alpha\gamma - \frac{1}{4}\beta\beta = -10 - \frac{25}{8} = -\frac{105}{8}$, $w - \frac{1}{4}\beta\gamma = -10 = \zeta$, $z - \frac{1}{4}\gamma\gamma = -5 - \frac{25}{64} = -\frac{345}{64} = \eta$, therefore, $2\delta$, $2\varepsilon$, $2\zeta$, $8\eta$, reſpectively, are $-5$, $-\frac{105}{4}$, $-20$, and $-\frac{345}{8}$, and their common Diviſor 5, which divided by 4, leaves 1, as it ought, becauſe the Term $s$ is odd. Since therefore, the common Diviſor $n$, or 5, is found, which gives Hope to a future Reduction, and becauſe it is odd, to $4z$, or $-20$, I ſucceſſively add $n$, $3n$, $5n$, $7n$, $9n$, &c. or 5, 15, 25, 35, 45, &c. and there ariſes $-15$, 0, 25, 60, 105, 160, 225, 300, 385, 480, 585, 700, 825, 960, 1105, 1260, 1425, 1600. of which only 0, 25, 225, and 1600, are Squares. Application of the foregoing rule to an example.

Wherefore, the Halves of thoſe Roots 0, $\frac{5}{2}$, $\frac{15}{2}$, 20 are to be collected in a Table for the Values of $S$, & the Values of $\sqrt{\frac{SS - z}{n}}$, that is, 1, $\frac{3}{2}$, $\frac{7}{2}$, 9 reſpectively for $b$. But becauſe $S + nb$, if 20 be taken for $S$, and 9 for $b$, becomes 65, a Number greater than four Times the greateſt Term of the Equation. Therefore I reject 20 and 9, and write only the reſt in the Table as follows, $b$ { 1. $\frac{3}{2}$, $\frac{7}{2}$, } $S$ { 0. $\frac{5}{2}$, $\frac{15}{2}$.

Then I try for $k$, all the Numbers which do not make $2 \pm 5k$ greater than 40, (four Times the greateſt Term of the Equation), that is, the Numbers $-8$, $-7$, $-6$, $-5$, $-4$, $-3$, $-2$, $-1$, 0, 1, 2, 3, 4, 5, 6, 7, putting $\frac{nkk + \alpha}{2}$ or $\frac{5kk - 5}{2}$, that is, the Numbers $\frac{315}{2}$, 120, $\frac{175}{2}$, 60, $\frac{75}{2}$, 20, $\frac{15}{2}$, 0, $-\frac{5}{2}$, 0, $\frac{15}{2}$, 20, $\frac{75}{2}$, 60, $\frac{175}{2}$, 120 reſpectively for $Q$: But ſince $Q \pm nl$, and much more $Q$ ought not to be greater than 40, I perceive I am to reject $\frac{315}{2}$, 120, $\frac{175}{2}$ and 60, and their Correſpondents $-8$, $-7$, $-6$, $-5$, 5, 6, 7 and conſequently that only $-4$, $-3$, $-2$, $-1$, 0, 1, 2, 3, 4 muſt reſpectively be tried for $k$,

and $\frac{75}{2}$, 20, $\frac{15}{2}$, 0, $-\frac{5}{2}$, 0, $\frac{15}{2}$, 20, $\frac{75}{2}$, respectively for $\mathcal{Q}$. Let us therefore try $-1$ for $k$, and 0 for $\mathcal{Q}$, and in this Case for $l$ there will be successively to be tried all the Numbers which do not make $\mathcal{Q} \pm nl$ greater than 40, that is, all the Numbers between 10 and $-10$; and for $R$ you are respectively to try the Numbers $\frac{2\beta - npkk}{4} + nkl$, or $-5 - 5l$, that is, $-55, -50, -45, -40, -35, -30, -25, -20, -15, -10, -5, 0, 5, 10, 15, 20, 25, 35, 40, 45$, the three former of which, and the last, because they are greater than 40, may be neglected. Let us try therefore, $-2$ for $l$, and 5 for $R$, and in this Case, for $m$ there will be besides to be tried all the Numbers which do not make $R \pm nm$, or $5 \pm 5m$ greater than 40, that is, all the Numbers between 7 and $-9$, and see whether or not, by putting $\frac{s - \mathcal{Q}\mathcal{Q} - pR + nll}{2}$ that is, $5 - 20 + 20$ or $5 = 2H$ $H + nkm$, or $\frac{5}{2} - 5m = S$, that is, if any of these Numbers $-\frac{65}{2}$, $-\frac{55}{2}$, $-\frac{45}{2}$, $-\frac{35}{2}$, $-\frac{25}{2}$, $-\frac{15}{2}$, $-\frac{5}{2}$, $\frac{5}{2}$, $\frac{15}{2}$, $\frac{25}{2}$, $\frac{35}{2}$, $\frac{45}{2}$, $\frac{55}{2}$, $\frac{65}{2}$, $\frac{75}{2}$, $\frac{85}{2}$, is equal to any of the Numbers 0, $\pm\frac{5}{2}$, $\pm\frac{15}{2}$, which were brought into the Catalogue for $S$, and we meet with four of these $-\frac{15}{2}$, $-\frac{5}{2}$, $\frac{5}{2}$, $\frac{15}{2}$ to which Answer $\pm\frac{7}{2}$, $\pm\frac{3}{2}$, $\pm\frac{3}{2}$, $\pm\frac{7}{2}$ written for $b$ in the same Table, & 2, 1, 0, $-1$ substituted for $m$, but let us try $-\frac{5}{2}$ or $S$, 1 for $m$, and $\pm\frac{3}{2}$ for $b$, and you will have $\frac{2RS - w}{2nm} = \frac{-25+10}{10} = -\frac{3}{2}$, and $\frac{2\mathcal{Q}S + RR - v - nmm}{2nl} = \frac{25+10-5}{-20} = -\frac{1}{2}$, and $\frac{pS + 2\mathcal{Q}R - t - 2nlm}{2nk} = \frac{-10+5+20}{-10} = -\frac{3}{2}$.

Wherefore, since there comes out in all Cases $-\frac{3}{2}$, or $b$, I conclude all the Numbers to be rightly found, and consequently that in the Room of the Equation proposed, we may write $x^4 + 2x^3 + 5x - 2\frac{1}{2} = \sqrt{5} \times (-x^3 - 2xx + x - 1\frac{1}{2})$. For by squaring the Parts of

this, there will be produced that Equation of eight Dimenſions which was at firſt propoſed.

CXX.

Reſolution of Equations of third and fourth degree.

Whatever Efforts the Analyſts have made, to find a general Method for ſolving Problems producing Equations of every Degree, beſides thoſe expreſſed by $ax^m = b$, and $ax^{2m} + bx^m = c$, they have not hitherto ſucceeded, except for ſuch as produce Equations of the third and fourth Degree, which we ſhall proceed to explain, beginning by the Reſolution of Problems, producing Equations of the third Degree.

CXXI.

*The Compound Intereſt of a certain Sum of Money, put out for* 4 *Years, amounted to a Sum* a, *but the ſimple Intereſt thereof for the ſame Time, and at the ſame Rate, would have been a Sum* b. *What was the Sum put out? And what the Rate of Intereſt?*

Let $y$ denote the Intereſt of 1 *l.* for one Year, therefore ſince the ſimple Intereſt of 1 *l.* for 4 Years, is $4y$, and the compound Intereſt $(1+y)^4 - 1$ or $4y + 6y^2 + 4y^3 + y^4$, we have, as $4y + 6y^2 + 4y^3 + y^4 : 4y = a : b$ by the Nature of the Queſtion, and conſequently $6y + 4y^2 + y^3 = \frac{4a - 4b}{b}$ from the Reſolution of which Equation, $y$ will be found, and conſequently the Rate of Intereſt required.

CXXII.

Equation the moſt complicated of the third degree.

Let in General the Equation to be ſolved, be $y^3 + dy^2 + ey + f = 0$, if we could reduce this Equation to another, including only the firſt and laſt Term, it is manifeſt that the Equation would be ſolved. Now the moſt natural Method of transforming an Equation into another, in which we are left at Liberty to make ſome Variation, is to ſubſtitute in this Equation in the Room of the unknown Quantity ſome other Quantity, in which a Letter is left undetermined in order to make uſe of it at pleaſure.

CXXIII.

Transformation by means of which any term is taken away from an Equation.

Let then $x + r$ be ſubſtituted in this Equation, in the Room of $y$, and there will reſult $x^3 + (3r + d)xx + (3rr + 2dr + e)x + r^3 + dr^2 + er + f = 0$. As $r$ in this Equation, is at our Diſpoſal, it is eaſy to perceive, that by its Means, any Term may be taken away out of the Equation.

Let, for Example, $3r + d$ be put $= 0$, or $r = -\frac{1}{3}d$, and the Equation is reduced to $x^3 + \left(e - \frac{d^2}{3}\right)x + \frac{2}{27}d^3 - \frac{de}{3} + f = 0$, the ſecond Term vaniſhing. If $3rr + 2dr + e$ be put $= 0$, or

$r = -\frac{1}{3}d \pm \sqrt{\left[-\frac{e}{3} + \frac{1}{9}dd\right]}$, the third Term will vaniſh, but the two others will remain.

It is eaſy to conceive, that by ſubſtituting $x + r$ in the Room of $y$, no more than one Term at a Time can be made to vaniſh, but if by this Tranſformation, we have not entirely effected the Reduction of the propoſed Equation to two Terms, at leaſt the Queſtion has been rendered more ſimple, ſince an Equation conſiſting only of three Terms remains to be ſolved.

Of the two transformed Equations that reſult by taking away either the 2d or 3d Term, the firſt, $x^3 + \left(e - \frac{d^2}{3}\right)x + \frac{2}{27}d^3 - \frac{de}{3} + f = 0$ is the moſt ſimple, which we ſhall therefore endeavour to reſolve, by ſtill trying to diminiſh the Number of its Terms: but before we proceed to this Reſearch, we ſhall explain how the Method employed for transforming an Equation of the third Degree, and exterminating its intermediate Terms, may be applied to Equations of every Degree.

CXXIV.

The foregoing transmation applied to an Equation of the fourth degree.

Let, for Example, the Equation propoſed be $y^4 + a\,y^3 + b\,y^2 + cy + d = 0$, the moſt general of the fourth Degree, ſubſtituting $x + r$ in the Room of $y$, there reſults.

$x^4 + (4\,r + a)\,x^3 + (6\,r^2 + 3\,a\,r + b)\,x^2 + (4\,r^3 + 3\,a\,r^2 + 2\,b\,r + c)\,x + r^4 + a\,r^3 + b\,r^2 + c\,r + d = 0$.

In which putting $4\,r + a = 0$, or $r = -\frac{1}{4}a$, there will reſult an Equation of the fourth Degree wanting the ſecond Term.

In like Manner, if $r$ be determined by Means of the Equation $6\,r^2 + 3\,a\,r + b = 0$, there will reſult an Equation of the fourth Degree, wanting the third Term, the fourth Term in like Manner may be made to vaniſh by ſolving and Equation of the 3d Degree, &c. But the Analyſts ſeldom or never take away any Term but the Second, becauſe the other Terms cannot be taken away without employing Calculations complicated with Radicals.

The ſecond term moſt commonly exterminated and why.

CXXV.

The ſecond term exterminated in an Equation of the fifth degree.

By ſubſtituting in the Equation $y^5 + a\,y^4 + b\,y^3 + cy^2 + dy + e = 0$, $x + r$ for $y$, there will reſult $x^5 + (5\,r + a)\,x^4 +$ &c. in which the ſecond Term will be taken away by ſolving the Equation $5\,r + a = 0$, or putting $r = -\frac{1}{5}a$: and in General, in an Equation of the $m$th Degree repreſented by $y^m + a\,y^{m-1} + b\,y^{m-2} +$ &c. $= 0$, if $y$ be put $= x + r$ it will be transformed into $x^m + (m\,r + a)\,x^{m-1} +$ &c. whereby it appears that the ſecond Term may be taken away from the propoſed Equation by putting $y = x - \frac{a}{m}$.

In an Equation of the $m$th degree.

CXXVI.

After this little Digreſſion, let us proceed to reſolve the Equation of the third Degree, $x^3 + \left(e - \frac{d^2}{3}\right)x + \left(\frac{2}{27}d^3 - \frac{de}{3} + f\right) = 0$, to which the general Equation $y^3 + dy^2 + ey + f = 0$, was reduced by putting $y = x - \frac{1}{3}d$, and to ſhorten the Calculation, let it be expreſſed thus, $x^3 + px + q = 0$.

Reſolution of the general Equation $x^3 + px + q = 0$.

Purſuing the ſame Method already employed, let the foregoing Equation be again transformed, by ſubſtituting, for Example, $u + z$ in the Room of $x$, not with a View to take away a Term of this Equation, as before, for it would ſoon appear that the Term taken away would return, but to reſolve this Equation into others more ſimple. Without perceiving that ſuch a Method would ſucceed; yet it is manifeſt, that the Transformation of an Equation into another, in which a Letter is left undetermined, cannot but be of Uſe.

$u + z$ being ſubſtituted for $x$, there reſults $u^3 + 3uuz + 3uzz + z^3 + pu + pz + q = 0$. Let one of the unknown Quantities $u$ or $z$ be ſuppoſed ſuch, that $u^3 + z^3 + q = 0$. In this Caſe we will have $3u^2z + 3uz^2 + pu + pz = 0$, which being divided by $u + z$, gives $3uz + p = 0$, or $u = -\frac{p}{3z}$. By which Equation one of the unknown Quantities introduced may be exterminated; to find the Equation that will determine the other unknown Quantity, let this Value of $u$ be ſubſtituted in the firſt Equation $u^3 + z^3 + q = 0$, and there will reſult

$-\frac{p^3}{27z^3} + z^3 + q = 0$, or $z^6 + qz^3 = \frac{p^3}{27}$, whence is deduced $z = \sqrt{\left[-\frac{1}{2}q \pm \sqrt{\left(\frac{1}{4}q^2 + \frac{1}{27}p^3\right)}\right]}$. Wherefore $u$ or $-\frac{p}{3z} = -\frac{\frac{1}{3}p}{\sqrt[3]{\left[-\frac{1}{2}q \pm \sqrt{\left(\frac{1}{4}q^2 + \frac{1}{27}p^3\right)}\right]}}$ which is reduced to $-\sqrt[3]{\left[\frac{1}{2}q \pm \sqrt{\left(\frac{1}{4}q^2 + \frac{1}{27}p^3\right)}\right]}$; for it is eaſy to perceive that the Product of $-\frac{1}{2}q \pm \sqrt{\left(\frac{1}{4}q^2 + \frac{1}{27}p^3\right)}$ into $+\frac{1}{2}q \pm \sqrt{\left(\frac{1}{4}q^2 + \frac{1}{27}p^3\right)}$ is $\frac{1}{27}p^3$ conſequently $\frac{1}{3}p$ is the Product of the cube Roots of thoſe Quantities. Now, adding together thoſe Values of $u$ and $z$, we will have $u + z$ or $x = \sqrt[3]{\left[-\frac{1}{2}q \pm \sqrt{\left(\frac{1}{4}q^2 + \frac{1}{27}p^3\right)}\right]} - \sqrt[3]{\left[+\frac{1}{2}q \pm \sqrt{\left(\frac{1}{4}q^2 + \frac{1}{27}p^3\right)}\right]}$, or $x = \sqrt[3]{\left[-\frac{1}{2}q + \sqrt{\left(\frac{1}{4}q^2 + \frac{1}{27}p^3\right)}\right]} - \sqrt[3]{\left[+\frac{1}{2}q + \sqrt{\left(\frac{1}{4}q^2 + \frac{1}{27}p^3\right)}\right]}$ for the Value of $x$ will be the ſame, whether the Radical $\sqrt{\left(\frac{1}{4}q^2 + \frac{1}{27}p^3\right)}$ be taken poſitively or negatively.

CXXVII.

The foregoing Formula expreſſes only one of the Values of the three Roots of the Equation, to find the two others we muſt divide the Equation $x^3 + px + q = 0$ by the Root found, and the Equation of the ſecond Degree that reſults for Quotient, will when reſolved, give the two other Roots required.

The foregoing formula expreſſes only one of the three Roots of the equation.

To find the general Expreſſion of theſe two Roots, I put, in order to abbreviate the Work $\sqrt[3]{[\frac{1}{2}q + \sqrt{(\frac{1}{4}qq + \frac{1}{27}p^3)}]} = m$ and $\sqrt[3]{[-\frac{1}{2}q + \sqrt{(\frac{1}{4}qq + \frac{1}{27}p^3)}]} = n$, then obſerving that $mn = \frac{1}{3}p$ and $m^3 - n^3 = q$, I divide the Equation $x^3 + px + q = 0$ by $x + m - n$, and there reſults the Equation $x^2 + nx - mx + mm + nn + mn = 0$ whoſe two Roots, are $x = \frac{m-n}{2} \pm \frac{1}{2} \times (m+n)\sqrt{-3}$, or $x = \frac{1}{2}\sqrt[3]{[\frac{1}{2}q + \sqrt[3]{(\frac{1}{4}qq + \frac{1}{27}p^3)}]} - \frac{1}{2}\sqrt[3]{[-\frac{1}{2}q + \sqrt[3]{\frac{1}{4}qq + \frac{1}{27}p^3)}]} \pm [\frac{1}{2}\sqrt[3]{[(\frac{1}{2}q + \sqrt{(\frac{1}{4}qq + \frac{1}{27}p^3)}]} + \frac{1}{2}\sqrt{[-\frac{1}{2}q + \sqrt{(\frac{1}{4}qq + \frac{1}{27}p^3)}]}] \times \sqrt{-3}$ which are neceſſarily imaginary when $\sqrt{(\frac{1}{4}q^2 + \frac{1}{27}p^3)}$ is a real Quantity.

Method of inveſtigating the two other roots.

CXXVIII.

The foregoing Solution of Equations of the third Degree is liable to this Inconveniency, that it gives no Value for $x$ when $\frac{1}{27}p^3$ is negative, and greater than $\frac{1}{4}qq$, in this Caſe, the Value of the Quantity $\sqrt{(\frac{1}{27}p^3 + \frac{1}{4}q^2)}$ is imaginary, whence the two Quantities $\sqrt[3]{[-\frac{1}{2}q + \sqrt{(\frac{1}{4}q^2 + \frac{1}{27}p^3)}]}$ and $-\sqrt[3]{[\frac{1}{2}q + \sqrt{(\frac{1}{4}q^2 + \frac{1}{27}p^3)}]}$ which compoſe the Value of $x$ are imaginary, we are not however to conclude, that the Value of $x$ in this Caſe is imaginary, it being real, as may be proved after the following Mannner.

Caſe in which the value of $x$ cannot be diſcovered by means of the foregoing formula on account of the imaginary quantities it includes.

Let the Value $\sqrt[3]{[-\frac{1}{2}q + \sqrt{(\frac{1}{4}q^2 + \frac{1}{27}p^3)}]} - \sqrt[3]{[\frac{1}{2}q + \sqrt{(\frac{1}{4}qq + \frac{1}{27}p^3)}]}$ of $x$ be reſumed, and let $a$ be put in the Room of $\frac{1}{2}q$ and $b\sqrt{-1}$ in the Room of $\sqrt{(\frac{1}{4}qq + \frac{1}{27}p^3)}$ ſuppoſed imaginary, and there will reſult $x = \sqrt[3]{(-a + b\sqrt{-1})} - \sqrt[3]{(a + b\sqrt{-1})}$. Now $\sqrt[3]{(-a + b\sqrt{-1})}$ and $-\sqrt{(a + b\sqrt{-1})}$ being reduced into Series by Means of the binomial Theorem,

The value of $x$ however in this caſe is real.

The firſt will become $-a^{\frac{1}{3}} + \frac{1}{3}a^{-\frac{2}{3}}b\sqrt{-1} - \frac{1}{9}a^{-\frac{5}{3}}b^2 - \frac{5}{81}a^{-\frac{8}{3}}b^3\sqrt{-1} + \frac{10}{243}a^{-\frac{11}{3}}b^4 + \&c.$

And the Second will become $-a^{\frac{1}{3}} - \frac{1}{3}a^{-\frac{1}{3}}b\sqrt{-1} - \frac{1}{9}a^{-\frac{5}{3}}b^2 + \frac{5}{81}a^{-\frac{8}{3}}b^3\sqrt{-1} + \frac{10}{243}a^{-\frac{11}{3}}b^4 - \&c.$ and $x$ will be =

$-2a^{\frac{1}{3}} - \frac{2}{3}a^{-\frac{5}{3}}b^2 + \frac{10}{243}a^{-\frac{11}{3}}b^4 - \frac{308}{6561}a^{-\frac{17}{3}}b^6 +$ &c. or

$-2a^{\frac{1}{3}} \times \left(1 + \frac{b^2}{9a^2} - \frac{10b^4}{243a^4} + \frac{154b^6}{6561a^6} -\right.$ &c.$\left.\right)$ Expreſſion entirely real.

Wherefore when $\frac{1}{27}p^3$ is negative and greater than $\frac{1}{4}qq$, the Value $\sqrt[3]{[-\frac{1}{2}q + \sqrt{(\frac{1}{27}p^3 + \frac{1}{4}q^2)}]} - \sqrt[3]{[\frac{1}{2}q + \sqrt{(\frac{1}{27}p^3 + \frac{1}{4}q^2)}]}$ of $x$, though exhibited under an imaginary Form, is however real. But the Analyſts have not been hitherto able to give it a real Form, without admitting an infinite Number of Terms in its Expreſſion.

CXXIX.

Method of approximating to the value of $x$ in this caſe.

Though the Value of $x$ cannot be determined exactly, we may however approximate to it by Means of the foregoing Series; for ſuppoſing $a$ greater than $b$, the Terms of this Series will decreaſe very ſwiftly, and a few of the firſt Terms will give a near Value of the Root required. But if $a$ is leſs than $b$, in finding the Value of $\sqrt[3]{(-a + b\sqrt{-1})}$ & $\sqrt[3]{(a + b\sqrt{-1})}$ by Means of the binomial Theorem, $b\sqrt{-1}$ muſt be put for the firſt Term of the Binomial, and $a$ for the Second, and there will reſult for the Value of $x$, or of $\sqrt[3]{(-a + b\sqrt{-1})} - \sqrt[3]{(a + b\sqrt{-1})}$, the Quantity

$$+\frac{2}{3}b^{-\frac{2}{3}}a - \frac{10}{81}b^{-\frac{8}{3}}a^3 + \frac{44}{729}b^{-\frac{14}{3}}a^5 - \frac{748}{19693}b^{-\frac{20}{3}}a^7 + \text{&c.}$$

or $-\frac{2a}{3b^{\frac{2}{3}}} \times -1 + \frac{5a^2}{27b^2} - \frac{22a^4}{243b^4} + \frac{374a^6}{6561b^6} -$ &c.

Expreſſion entirely Real, and whoſe Terms decreaſe very ſwiftly, when $a$ is leſs than $b$.

CXXX.

The two other values of $x$, are real in the ſame caſe.

Of the three Roots of the Equation $x^3 + px + q = 0$, thoſe $x = \frac{1}{2}\sqrt[3]{[\frac{1}{2}q + \sqrt{(\frac{1}{4}qq + \frac{1}{27}p^3)}]} - \frac{1}{2}\sqrt[3]{[\frac{1}{2}q + \sqrt{(\frac{1}{4}qq + \frac{1}{27}p^3)}]} \pm [\frac{1}{2}\sqrt{[\frac{1}{2}q + \sqrt{(\frac{1}{4}q^2 + \frac{1}{27}p^3)}]} + \frac{1}{2}\sqrt{[-\frac{1}{2}q + \sqrt{(\frac{1}{4}qq + \frac{1}{27}p^3)}]}] \times \sqrt{-3}$ are both imaginary, when $\sqrt{(\frac{1}{4}qq + \frac{1}{27}p^3)}$ is a real Quantity. We ſhall now ſhew that theſe two Roots are real when $\sqrt{(\frac{1}{4}qq + \frac{1}{27}p^3)}$ is imaginary, that is, when $p$ is negative, and $\frac{1}{27}p^3$ is greater than $\frac{1}{4}qq$.

For changing by Means of the Denominations already employed, the Expreſſion of thoſe two Values of $y$ into $\frac{1}{2}\sqrt[3]{(-a + b\sqrt{-1})} - \frac{1}{2}\sqrt[3]{(a + b\sqrt{-1})} \pm [\frac{1}{2}\sqrt[3]{(a + b\sqrt{-1})} + \frac{1}{2}\sqrt[3]{(-a + b\sqrt{-1})}] \times \sqrt{-3}$ and reducing it into Series, there will reſult for the the two Values of $x$.

$$a^{\frac{2}{3}} \times 1 + \frac{bb}{9aa} - \frac{10\,b^4}{243\,a^4} + \frac{154\,b^6}{6561\,a^5} \text{ &c.} \pm \frac{b}{a^{\frac{2}{3}}\sqrt{3}} \times 1 - \frac{5\,bb}{27\,aa}$$

$$+ \frac{22\,b^4}{243\,a^4} - \frac{374\,b^6}{6561\,a^6} + \text{ &c. Expreſſion entirely real.}$$

CXXXI.

How from the roots of the Equation $x^3+px+q=0$ are deduced the roots of the equation $y^3+dy^2+ey+f=0$.

As the moſt general Equation of the third Degree repreſented by $y^3 + dy^2 + ey + f = 0$ is reduced to $x^3 + \left(e - \frac{d^2}{3}\right)x + \frac{1}{27}d^3 - \frac{de}{3} + f = 0$, or to $x^3 + px + q = 0$, by ſuppoſing $y = x - \frac{1}{3}d$ it follows that the Values of $y$ in the general Equation $y^3 + dy^2 + ey + f = 0$ will be found by reſolving this laſt Equation $x^3 + px + q = 0$, and deducting $\frac{1}{3}d$ from its three Roots.

CXXXII.

An Equation of the third degree has three real roots, or one real root and two impoſſible roots.

How to diſtinguiſh thoſe caſes.

Whence it appears, that every Equation of third Degree has at leaſt one real Root, and that the two others are either both real or imaginary.

To diſcover which of thoſe two Caſes happens in any propoſed Equation of the third Degree, the ſecond Term muſt be exterminated, in order to compare it with the Equation $x^3 + px + q = 0$; which being done, if $p$ be poſitive, or if negative, $\frac{1}{27}p^3$ is not greater than $\frac{1}{4}qq$, the Equation will have only one real Root expreſſed by the Formula of Art. CXXVI. But if $p$ is negative, and $\frac{1}{27}p^3$ is greater than $\frac{1}{4}qq$, the three Roots will be real, but cannot otherwiſe be determined than by Approximation.

Nature of the roots when $\frac{1}{27}p^3$ is negative and equal $\frac{1}{4}qq$.

If $\frac{1}{27}p^3$ is negative and equal to $\frac{1}{4}qq$ the firſt Value of $x$ expreſſed by $\sqrt[3]{\left[-\frac{1}{2}q + \sqrt{\left(\frac{1}{4}q^2 + \frac{1}{27}p^3\right)}\right]} - \sqrt[3]{\left[\frac{1}{2}q + \sqrt{\left(\frac{1}{4}q^2 + \frac{1}{27}p^3\right)}\right]}$ is reduced to $-2\sqrt[3]{\frac{1}{2}q}$ and the two other Values expreſſed in general by $\frac{1}{2}\sqrt[3]{\left[\frac{1}{2}q + \sqrt{\left(\frac{1}{4}qq + \frac{1}{27}p^3\right)}\right]} - \frac{1}{2}\sqrt[3]{\left[-\frac{1}{2}q + \sqrt{\left(\frac{1}{4}qq + \frac{1}{27}p^3\right)}\right]} \pm \left[\frac{1}{2}\sqrt[3]{\left[\frac{1}{2}q + \sqrt{\left(\frac{1}{4}qq + \frac{1}{27}p^3\right)}\right]} + \frac{1}{2}\sqrt[3]{\left[-\frac{1}{2}q + \sqrt{\left(\frac{1}{4}qq + \frac{1}{27}p^3\right)}\right]}\right] \times \sqrt{-3}$ is reduced to $+\sqrt[3]{\frac{1}{2}q}$. Conſequently in this Caſe the three Roots of the Equation are real.

CXXXIII.

Application of the foregoing Methods to the reſolution of the propoſed problem.

To ſolve the Equation $y^3 + 4y^2 + 6y = \frac{4a - 4b}{b}$ produced by the foregoing Problem: Let $a = 344{,}81\ l$; $b = 320\ l$; whereby this Equation is transformed into $y^3 + 4y^2 + 6y = 0{,}310125\text{r}$. I firſt exterminate the ſecond Term by putting $y = x - \frac{4}{3}$, and by this Subſtitution there reſults the Equation $x^3 + 0{,}666\,x - 3{,}569384 = 0$; which compared with $x^3 + px + q = 0$, gives $p = 0{,}666$ &c. and $q = -3{,}569384$: thoſe Values being ſubſtituted in $\sqrt[3]{\left(\frac{1}{4}qq + \frac{1}{27}p^3\right)}$, this Quantity becomes

$\sqrt[3]{}$ 3,19538 which is a real Quantity, whence the Value required of $x$ will be obtained by Means of the Formula of Art. CXXVI. Substituting therefore in $\sqrt[3]{[-\frac{1}{2}q+\sqrt{(\frac{1}{4}qq+\frac{1}{27}p^3)}]}-\sqrt[3]{[\frac{1}{2}q+\sqrt{(\frac{1}{4}qq+\frac{1}{27}p^3}]}$, $-1,784692$ for $\frac{1}{2}q$, and $\sqrt{}$ 3,19538 for $\sqrt{(\frac{1}{4}qq+\frac{1}{27}p^3)}$; this Value of $x$ becomes $\sqrt[3]{}$ 1,784692 $+\sqrt{}$ 3,19538 $-\sqrt[3]{}$ $-1,784692+\sqrt{}$ 3,19538 and substituting this Value of $x$ in $y=x-1,333$, &c. there results $y=\sqrt[3]{}[1,784692+\sqrt{}3,19538]-\sqrt[3]{}[-1,784692+\sqrt{}3,19538]-1,333$ $=,05$, which is the only real Root of the Equation, therefore the Rate of Interest was 5 *per Cent.* and the Sum put out 1600*l.*

CXXXIV.

Every Equation of the sixth Degree, in which the Dimensions of the unknown Quantity are all even, may be resolved by the foregoing Method. Let for Example, the Equation $z^6+9z^4+39zz+55=0$, be proposed, putting $zz$ equal to a new unknown Quantity $x$ less the third Part of the Coefficient of the second Term, that is, putting $zz=x-3$, the Equation will be transformed into $x^3+12x-8=0$, which is only of the third Degree, and wants the second Term.

Equation of the sixth degree reduced to one of the third.

Comparing now this Equation with $x^3+px+q=0$, we have $p=12$, $q=-8$, and consequently $\sqrt{(\frac{1}{27}p^3+\frac{1}{4}qq)}=\sqrt{}80$; whence $x$ or $\sqrt[3]{-\frac{1}{2}q+\sqrt{(\frac{1}{27}p^3+\frac{1}{4}q^2)}}]-\sqrt[3]{[\frac{1}{2}q+\sqrt{(\frac{1}{27}p^3+\frac{1}{4}qq)}]}$ $=\sqrt[3]{(4+\sqrt{}80)}-\sqrt[3]{(\sqrt{}80-4)}$, and since by Supposition $zz=x-3$, or $z=\sqrt{(x-3)}$, we will have the unknown Quantity sought $x=\pm\sqrt{[\sqrt[3]{(4+\sqrt{}80)}-\sqrt[3]{\sqrt{(80-4)}-3}]}$ and the two Values given by this Expression are the only real Ones of the Six that the proposed Equation admits of.

Equation of a higher degree reduced in like manner to one of the third.

In general, all Equations as $z^{3m}+az^{2m}+bz^m+c=0$, is reduced to one of the third Degree wanting the second Term, by putting $z^m=x-\frac{1}{3}a$.

CXXXV.

*Let it be proposed to divide the Number* 24 *into two such Parts, that the Difference of their Cubes may be* 3584.

Let $12+x$ express the greater Part, and $12-x$ the Lesser, then will $(12+x)^3-(12-x)^3=3584$. That is, $x^3+432x=1792$, which being compared with $x^3+px+q=0$, I find $p=432$, $q=-1792$, and as $p$ is positive, I conclude Art. CXXXII. that the Formula of Art. CXXVI will succeed in this Case, substituting in Effect the Values of $p$ and of $q$ in the general Formula $x=\sqrt[3]{[-\frac{1}{2}q+\sqrt{(\frac{1}{27}p^3+\frac{1}{4}qq}]}-\sqrt[3]{[\frac{1}{2}q+\sqrt{(\frac{1}{27}p^3+\frac{1}{4}qq)}]}$ we will have $x=\sqrt[3]{}896+\sqrt{}3788800-\sqrt[3]{}-896+\sqrt{}3788800$, which by the Rule of Art. LXXXVIII. will be found to be reducible, since $896+\sqrt{}3788800$ is the Cube of $2+\sqrt{}148$ and $-896+\sqrt{}3788800$ is the Cube of $-2+\sqrt{}148$, whence the Value of $x$ is reduced to 4, whence 8 and 16 are the two required Numbers.

Another problem producing an Equation of the third degree solved by the foregoing method.

CXXXVI.

*What two Numbers are those, whose Product is* 98, *and if from the Square of the greater, the lesser be taken, the Remainder will be* 90.

Problem producing an Equation of the third degree that cannot be solved by the formula of Art. cxxvi.

Suppose $x$ the Greatest, and $y$ the least of those Numbers, then $xy = 98$, or $y = \frac{98}{x}$, & $xx - y = 90$, or $y = xx - 90 = \frac{98}{x}$ or $x^3 - 90x = 98$. Comparing this Equation with the general Equation $x^3 + px + q = 0$, we have $p = -90$, $q = -98$. Now $p$ being negative, and $\frac{1}{27}p^3$ being greater than $\frac{1}{4}qq$, the Equation cannot be resolved by the foregoing Formula. I therefore employ the Method of Art. cxxix. to find an approximate Value of $x$.

To this End, I substitute for $p$ and $q$ their Values $-90$, and $-98$, in $x = \sqrt[3]{[-\frac{1}{2}q + \sqrt{(\frac{1}{27}p^3 + \frac{1}{4}qq)}]} - \sqrt[3]{[\frac{1}{2}q + \sqrt{(\frac{1}{27}p^3 + \frac{1}{4}qq)}]}$ and there Results $x = \sqrt[3]{[49 + \sqrt{-24599}]} - \sqrt[3]{[-49 + \sqrt{(-24599)}]}$ which compared with $\sqrt[3]{(-a + b\sqrt{-1})} - \sqrt[3]{(a + b\sqrt{-1})}$ or $-\frac{2a}{3b^{\frac{2}{3}}} \times -1 + \frac{5a^2}{27b^2} - \frac{22a^4}{243b^4} + \frac{274a^6}{6561b^6}$ &c. gives $a = -49$ $bb = +24599$, which are to be substituted in this infinite Series.

Application of the method of Art. cxxix. for approximating to the roots of the Equation produced by the foregoing problem.

To perform this Substitution, I first extract the cube Root of 24599 to obtain $b^{\frac{2}{3}}$, which I find to be nearly 29,08, and consequently $\frac{2a}{3b^{\frac{2}{3}}}$ or $\frac{98}{3\sqrt{24599}}$ is nearly $\frac{10000}{8902}$.

Squaring afterwards $a$, and dividing it by $bb$, I find $\frac{aa}{bb}$ to be nearly 0,0976, whose Square 0,00952 is the Value of $\frac{a^4}{b^4}$; as to the Value of $\frac{a^6}{b^6}$ and of the higher Powers, they are to be disregarded in this Series, which converges very swiftly. Substituting those Values of $\frac{a^2}{b^2}$, $\frac{a^4}{b^4}$ in the infinite Series, I find $-1,104$ for the Value of $x$ given by the Formula of Art. cxxvi. to obtain the other two Values of $x$, which are also real, Art. cxxxii. I divide the Equation $x^3 - 90x - 98 = 0$, by $x + 1,104$ and there results for Quotient $xx - 1,104x - 88,781$ with a Remainder 0,0142, which may be rejected as exceeding small. So that $xx - 1,104x - 88,781 = 0$, may be considered as the exact Quotient arising from the Division of $x^3 - 90x - 98 = 0$ by $x + 1,104$, and as the Product of the two Roots sought; resolving therefore this Equation, I find $x = 0,552 + \sqrt{89,0857}$, that is, $+9,990$ and $-8,886$, consequently, the three Values of $x$ in the proposed Equation $x^3 - 90x - 98 = 0$,

are $-1,104$; $+9,990$; $-8,886$, whereof the ſecond only is for our Purpoſe.

Inconveniency to which the foregoing Method of approximating to the roots of Equations of the third degree is liable.

The foregoing Method of reſolving by Approximation, Equations of the third Degree, whoſe three Roots are real, is liable to this Inconveniency, that when $a$ differs but little from $b$, the Terms of the Series expreſſing the Value of $x$, decreaſe ſo ſlowly, that a great Number will be required to give a near Value of the Root, whereby the Computation is rendered extreamly laborious. To remedy which, the Analyſts have ſought a Method in general more commodious in Practice, which we ſhall proceed to explain.

CXXXVII.

With this View, let the Equation $x^3 + px + q = 0$, or rather $x^3 - px + q = 0$, (the Caſe in which the Method of Art. CXXVI. fails, being comprehended among thoſe in which $p$ is negative) be reſumed, which for greater Simplicity I propoſe to reduce to this Form, $z^3 - z = r$, to effect which, I put $x = mz$, which transforms the Equation $x^3 - px + q = 0$ into $z^3 - \frac{p}{mm}z = -\frac{q}{m^3}$, whence it appears, that by putting $m = -\sqrt{p}$, if $q$ is poſitive, and $m = \sqrt{p}$, if $q$ is negative, the Equation $x^3 - px + q = 0$ will be reduced to the Form $z^3 - z = r$.

Another general method of approximation more commodious in practice.

I now obſerve, that if this Equation be of the Number of thoſe that cannot be ſolved by the Formula of Art. CXXVI. $r$ muſt be leſs than $\frac{1}{3}\sqrt{\frac{4}{3}}$, or $\sqrt{\frac{4}{27}}$, and at the ſame Time one of the Roots of the Equation muſt be poſitive, and greater than Unity, but leſs than $\sqrt{\frac{4}{3}}$, for if $z$ exceeded $\sqrt{\frac{4}{3}}$, there would reſult for $r$, that is, for $z^3 - z$ a Number greater than $\sqrt{\frac{4}{27}}$, and conſequently the Equation $z^3 - z = r$, would be of the Number of thoſe that can be ſolved by Means of the Formula of Art. CXXVII.

This being premiſed, I put $z = 1 + \delta$, and ſubſtituting this Value in the Equation $z^3 - z = r$, there reſults $-r + 2\delta + 3\delta\delta + \delta^3 = 0$, which multiplied by four Terms of the Series $1 + A\delta + B\delta\delta + C\delta^3 +$ &c. the Product will be

$$-r + (2 - rA)\delta + (3 + 2A - rB)\delta\delta + (1 + 3A + 2B - rC)\delta^3 + (A + 3B + 2C)\delta^4 + (B + 3C)\delta^5 + C\delta^6,$$

in which the third, fourth and fifth Terms may be made to vaniſh, by determining $A$, $B$, and $C$ by Means of the Equations $3 + 2A - rB = 0$, $1 + 3A + 2B - rC = 0$, $A + 3B + 2C = 0$, whence is deduced $B = \frac{2A}{r} + \frac{3}{r}$, $C = \frac{2B}{r} + \frac{3A}{r} + \frac{1}{r}$, $0 = 2C + 3B + A$, whence exterminating $C$, $\frac{2B}{r} + \frac{3A}{r} + \frac{1}{r} + \frac{3B}{2} + \frac{A}{2} = 0$, or $\left[\frac{2}{r} + \frac{3}{2}\right] \times B + \left[\frac{3}{r} + \frac{1}{2}\right] \times A + \frac{1}{r} = 0$,

which by substituting the Value of $B$ becomes

$\left[\frac{2}{r}+\frac{3}{2}\right]\times\left[\frac{2A}{r}+\frac{3}{r}\right]+\left[\frac{3}{r}+\frac{1}{2}\right]\times A+\frac{1}{r}=0$, or,

$\left[\frac{4}{r^2}+\frac{6}{r}+\frac{1}{2}\right]\times A+\frac{6}{r^2}+\frac{9}{2r}+\frac{1}{r}=0$, multiplying by $2r^2$,

$(8+12r+r^2)\times A+12+9r+2r=0$, whence $A$ is found

$=-\frac{12+11r}{8+12r+r^2}$, $B=\frac{14+3r}{8+12r+r^2}$, $C=-\frac{-15+r}{8+12r+r^2}$,

substituting those Values in the foregoing Equation there results

$-r+\left[\frac{16+36r+13r^2}{8+12r+r^2}\right]\delta-\left[\frac{31-6r}{8+12r+r}\right]\delta^5-\frac{15-r}{8+12r+r^2}\delta^6=0$

which by rejecting the two last Terms as exceeding small, is reduced to,

$+8r+12r^2+r^3=(16+36r+13r^2)\times\delta$; hence

$\delta=\frac{8r+12r^2+r^3}{16+36r+13r^2}$, Consequently $1+\delta$ or $z=\frac{16+44r+25r^2+r^3}{16+36r+13r^2}$.

CXXXVIII.

To discover the Error produced in the Value of $z$ by rejecting the Terms $-\frac{31-6r}{8+12r+r^2}\delta^5-\frac{15-r}{8+12r+r^2}\delta^6$, it suffices to try that Case in which those Terms are greatest, let then $z$ or $1+\delta$ be supposed to be equal to $\sqrt{\frac{4}{3}}$, which is greater than can ever happen, when the proposed Equation is of the Number of those that cannot be solved by the Method of Art. CXXVI. in this Case $r=\frac{1}{3}\sqrt{\frac{4}{3}}$, and the foregoing Method will give for the Value of $z$, $\frac{1596+1192\sqrt{\frac{4}{3}}}{1452+972\sqrt{\frac{4}{3}}}$ instead of the true Value $\sqrt{\frac{4}{3}}$, now it is easy to perceive that these two Quantities differ only by the one ten thousanth Part of an Unit, and this is consequently the greatest Error that can be committed by taking $\frac{16+44r+25r^2+r^3}{16+36r+13r^2}$ for the positive Root of the Equation $z^2-z=r$, if this Equation be of the Number of those to which the Formula of Art. CXXVI. cannot be applied,

The foregoing method of approximation give the value of $x$ true to the fifth decimal place.

After the Value of $z$ has been computed by the Expression $\frac{16+44r+25r^2+r^3}{16+36r+13r^2}$, it must be multiplied by $m$, to obtain an approximate Value of $x$.

CXXXIX.

If ſtill a greater Degree of Exactneſs is required, more Terms of the Series $1 + Az + Bz^2 + Cz^3$ &c. muſt be taken in. Now, it is evident that in all Caſes, $B = \frac{2A}{r} + \frac{3}{r}$, $C = \frac{2B}{r} + \frac{3A}{r} + \frac{1}{r}$; $D = \frac{2C}{r} + \frac{3B}{r} + \frac{A}{r}$; $E = \frac{2D}{r} + \frac{3C}{r} + \frac{B}{r}$, &c. where the laſt Value is always equal to nothing.

Method of approximating to the value of $z$ to any aſſigned degree of exactneſs.

Subſtituting in the firſt of theſe Equations $B = \frac{2A}{r} + \frac{3}{r}$, $Q$ and $q$ for $\frac{2}{r}$ and $\frac{3}{r}$ reſpectively, it becomes $B = QA + q$, this Value of $B$ wrote in the ſecond Equation $C = \frac{2B}{r} + \frac{3A}{r} + \frac{1}{r}$, gives $C = \frac{2QA}{r} + \frac{2q}{r} + \frac{3A}{r} + \frac{1}{r}$, which by ſubſtituting $R$ and $r'$, for $\frac{2Q+3}{r}$ an $\frac{2q+1}{r}$ reſpectively, becomes $RA + r'$.

Thoſe Values of $B$ and $C$ ſubſtituted in the third Equation, gives $D = \frac{2RA}{r} + \frac{2r'}{r} + \frac{3QA}{r} + \frac{3q}{r} + \frac{A}{r}$, which by ſubſtituting $S$ and $s$ for $\frac{2R+3Q+1}{r}$ and $\frac{2r'+3q}{r}$ reſpectively, becomes $SA + s$.

After the ſame Manner the fourth Equation will be reduced to $E = TA + t$, by ſubſtituting $T$ and $t$ for $\frac{2S+3R+Q}{r}$, $\frac{2s+3r'+q}{r}$ reſpectively, &c.

Whence is derived the following Rule for obtaining an Approximation to take in as many Terms of the propoſed Series as you pleaſe. Put

$Q = \frac{2}{r}$, $R = \frac{2Q+3}{r}$; $S = \frac{2R+3Q+1}{r}$; $T = \frac{2S+3R+Q}{r}$; $V = \frac{2T+3S+R}{r}$, &c.

$q = \frac{3}{r}$, $r' = \frac{2r+1}{r}$, $s = \frac{2r'+3q}{r}$, $t = \frac{2s+3r'+q}{r}$, $v = \frac{2s+3s+r'}{r}$

Divide the laſt of theſe Quantities $q$, $r$, $s$, $t$, &c. by the correſponding

Quantity of the upper Series $\mathfrak{Q}$, $R$, $S$, &c. then $\frac{r}{2+r\times\frac{1}{\mathfrak{Q}}}$, $\frac{r}{2+r\times\frac{r}{R}}$, $\frac{r}{2+r\times\frac{s}{S}}$ &c. or $\frac{1}{\frac{2}{r}+\frac{q}{\mathfrak{Q}}}$, $\frac{1}{\frac{2}{r}+\frac{r'}{R}}$, $\frac{1}{\frac{2}{r}+\frac{s}{S}}$, $\frac{1}{\frac{2}{r}+\frac{t}{T}}$, $\frac{1}{\frac{2}{r}+\frac{v}{V}}$, &c. will be so many different Values of $\delta$, whereof each is more exact than the preceding One.

CXL.

Application of this Method to an Example.

To shew the Application of the foregoing Approximations by an Example, let the Equation $z^3 - z = \frac{1}{3}$ be proposed.

Here $\frac{2}{r}=6$, $\frac{3}{r}=9$, $\frac{1}{r}=3$, Hence $\mathfrak{Q}=\frac{2}{r}=6$, $R=\frac{2\mathfrak{Q}}{r}+\frac{3}{r}=45$, $S=\frac{2R}{r}+\frac{3\mathfrak{Q}}{r}+\frac{1}{r}=327$, $T=\frac{2S+3R+\mathfrak{Q}}{r}=2385$, $V=\frac{2T+3S+R}{r}=17388$, Also $q=\frac{3}{r}=9$, $r'=\frac{3q+1}{r}=57$, $s=\frac{2r'+3q}{r}=423$, $t=\frac{2s+3r'+q}{r}=3078$, $v=\frac{2t+3s+r'}{r}=22446$, Therfore $\frac{q}{\mathfrak{Q}}=\frac{9}{6}$, $\frac{r'}{R}=\frac{57}{45}$, $\frac{s}{S}=\frac{423}{327}$, $\frac{t}{T}=\frac{3078}{2385}$ $\frac{v}{V}=\frac{22446}{17388}$. From whence $\delta=0,137158$, and consequently $z=1,137158$.

CXLI.

*There are three Numbers in geometrical Progression whose Sum is a, and the Sum of their Cubes* b; *what are those Numbers.*

Problem producing an Equation of the third degree solved by this method.

If $x$, $y$ and $z$ denote the Numbers required, then by the Question $x+y+z=a$, $x^2+y^2+z^2=b$, or by Transposition $x+z=a-y$ and $x^2+z^2=b-y^2$ & $(x+z)^2=(a-y)^2$. Now the Difference of those two last Equations is $3x^2z+3xz^2=a^2-3a^2y+3ay^2-b$, but $x+z=a-y$ and $xz=y^2$, whence $y^3-a^2y+\frac{a^2-b}{3}=0$.

Let $a$ and $b$ be such that $a^2=13$, & $\frac{a^2-b}{3}=5$. Then will the Equation be transformed into $y^3-13y+5=0$, and substituting $-z\sqrt{13}$

for $y$, it will become $z^8 - z = \frac{5}{13\sqrt{13}}$, whence $z$ is deduced, and conſequently $y = -3{,}784341$, &c.

CXLII.

Having ſhewn how to ſolve Problems producing Equations of the third Degree, we ſhall proceed to explain the Methods employed by the Analyſts for ſolving Problems producing Equations of the fourth Degree, of which the following is one of the moſt general of this Order.

*The Sum* a, *the Sum of the Squares* b, *the Sum of the Cubes* c, *and the Sum of the Biquadrates* d, *of any four Numbers being given; to determine the Numbers.*

Problem producing an Equation the moſt compliecated of the fourth degree.

Let the four Numbers be denoted by $x$, $y$, $z$ and $u$, and put $A = a - u$, $B = b - u^2$, $C = c - u^3$, and $D = d - u^4$. From whence by the Conditions of the Problem.

$x + y + z = A$, $x^2 + y^2 + z^2 = B$, $x^3 + y^3 + z^3 = C$, $x^4 + y^4 + z^4 = D$

Now, if the ſecond of theſe Equations be ſubſtracted from the Square of the firſt, we ſhall have $2xy + 2xz + 2yz = A^2 - B$.

And if, in the like Manner, the fourth Equation be ſubtracted from the Square of the 2d, we ſhall have, $2x^2y^2 + 2x^2z^2 + 2y^2z^2 = B^2 - D$.

Moreover, if from the Square of the former of theſe laſt Equations, the Double of the Latter be deducted, there will come out

$8x^2yz + 8y^2xz + 8z^2xy = A^4 - 2A^2B - B^2 + 2D$;

or, $8xyz \times (x + y + z) = A^4 - 2A^2B - B^2 + 2D$;

whence, $xyz = \frac{A^4 - 2A^2B - B^2 + 2D}{8A}$ (becauſe $x + y + z = A$)

Again, by multiplying the firſt and fifth Equations into each other, we get $2x^2y + 2x^2z + 2y^2x + 2y^2z + 2z^2x + 2z^2y + 6xyz = A^3 - AB$, and by multiplying the firſt and third, Equations together, there ariſes $x^3 + y^3 + z^3 + x^2y + x^2z + y^2x + y^2z + z^2x + z^2y = AB$.

The Double of which laſt taken from the Precedent, leaves $6xyz - 2x^3 - 2y^3 - 2z^3 = A^3 - 3AB$, and this added to $2x^3 + 2y^3 + 2z^3 = 2C$, gives $6xyz = A^3 - 3AB + 2C$.

Hence $\frac{A^3 - 3AB + 2C}{6} = xyz = \frac{A^4 - 2A^2B - B^2 - 2D}{8A}$

and conſequently by Reduction $A^4 - 6A^2B + 8AC + 3B^2 - 6D = 0$.

In which Equation let the ſeveral Values of $A$, $B$, $C$ and $D$ be

now ſubſtituted, and dividing the Whole by 24, we at Length, have

$$u^4 - a u^3 + \frac{a^2 - b}{2} \times u^2 - \frac{a^3 - 3ab + 2c}{6} \times u + \frac{a^4 - 6a^2 b + 8ac + 3b^2 - 6d}{24} = 0.$$ Whoſe Roots Anſwer all the Conditions of the Problem.

CXLIII.

Reſolution of the general Equation of the fourth degree.

Let in general $u^4 + a u^3 + b u^2 + c u + d = 0$ repreſent all Equations of the fourth Degree, the Difficulty is ſoon reduced to ſolve an Equation repreſented by $z^4 + p z^2 + q z + r = 0$ by putting $u = z - \frac{1}{4} a$.

Now, the Manner the moſt natural of attempting the Reſolution of this Equation, is to conſider it as the Product of two Equations of the ſecond Degree, aud ſo contrive that the Determination of the Coeficients affecting the Terms of thoſe Equations of the ſecond Degree, depend on Equations eaſier to be ſolved than the propoſed one.

Let firſt $zz + xz + s = 0$ be taken for one of thoſe Equations; it is manifeſt that the other ſhould have for ſecond Term $- xz$, ſince the Product of thoſe two Equations ſhould give an Equation wanting the ſecond Term. Let then $zz - xz + t = 0$, be taken for this ſecond Equation, multiplying thoſe two Equations into one another there will reſult.

$z^4 + (s - x^2 + t) zz + (sx - tx) z + st = 0$, which being compared with the propoſed Equation, gives for determining $s$, $t$, $x$ the three Equations $s - x^2 + t = p$; $sx - tx = q$; $st = r$.

To make Uſe of thoſe three Equations, I multiply the firſt by $x$, and then add it to the ſecond Equation, and there reſults $2sx - x^3 = px + q$, whence is deduced $s = \frac{q + px + x^3}{2x}$, which ſubſtituted in the Equation $st = r$ gives $t = \frac{2rx}{x^3 + px + q}$, and ſubſtituting thoſe two Values of $s$ and $t$ in the Equation $sx - tx = q$, there reſults at Length $\frac{x^3 + px + q}{2} - \frac{2rx^2}{x^3 + px + q} = q$, or, $x^6 + 2px^4 + (pp - 4r)x^2 - q^2 = 0$,

The Reſolution of an Equation of the third degree depends on one of the third degree which is called by the Analyſts the reduced Equation.

Equation of the ſixth Degree, which by the Method of Art. CXXXIV, is tranſformed into one of the Third, whence the Difficulty of Equations of the fourth Degree is reduced to that of the Third, for this laſt Equation (which is called the reduced) being ſolved, we have no more to do but to ſubſtitute the Value it gives for $x$, in the Equations $zz - xz + s = 0$, $zz + xz + t = 0$,

or rather in $zz - xz + \frac{1}{2}xx + \frac{1}{2}p + \frac{q}{2x} = 0$ and in $zz + xz + \frac{2r}{xx + p + \frac{q}{x}} = 0$, and afterwards ſolve thoſe Equations,

or what amounts to the ſame, ſubſtitute the Value of $x$ in the Roots $z = \frac{1}{2}x \pm \sqrt{\left[-\frac{1}{4}xx - \frac{1}{2}p - \frac{q}{2x}\right]}$, & $z = -\frac{1}{2}x \pm \sqrt{\left[-\frac{2r}{x^2 + p + \frac{q}{x}} + \frac{1}{4}xx\right]}$

of thoſe two Equations, and the four Roots required of the Equation $z^4 + pz^2 + qz + r = 0$, and conſequently of the propoſed Equation, $u^4 + au^3 + bu^2 + cu + d = 0$ will be obtained.

CXLIV.

The four Roots of an Equation of the fourth degree may be expreſſed by one formula.

It appears at firſt View, from the Expreſſion of thoſe Values, that in the fourth Degree, as in the Third, one Expreſſion cannot be found for all the Roots of the Equation. However, when it is obſerved that the Quantity $x$ included in the two foregoing Expreſſions is neceſſarily a ſquare Radical, ſince it reſulted from the Reſolution of an Equation in which $x$ is always of an even Number of Dimenſions, it will eaſily appear that each of thoſe Expreſſions may denote four Roots, the firſt being then expreſſed thus, $z = \pm\frac{1}{2}x \pm \sqrt{\left(-\frac{1}{4}xx - \frac{1}{2}p \mp \frac{q}{2x}\right)}$ and the Second thus, $z = \mp\frac{1}{2}x + \sqrt{\left(\frac{1}{4}xx - \frac{2r}{xx + p \pm \frac{q}{x}}\right)}$.

Thoſe two Equations in Appearance differing; may render the foregoing Reaſoning ſuſpect, ſince it ſeems to lead to this Abſurdity, that an Equation of the fourth Degree may have eight Roots, but this Difference is only apparent, for the Identity of thoſe two Expreſſions is reduced to that of $\sqrt{\left(\frac{1}{4}xx - \frac{2r}{xx + p \mp \frac{q}{x}}\right)}$ and of $\sqrt{\left(-\frac{1}{4}xx - \frac{1}{2}p \mp \frac{q}{2x}\right)}$, that is, of $-\frac{1}{2}xx - \frac{1}{2}p \mp \frac{q}{2x}$, and of $-\frac{2r}{xx + p \mp \frac{q}{x}}$. But the Identity of thoſe two laſt Quantities cannot fail to take Place, when $x$ has been determined by Means of the Equation $x^6 + 2px^4 + (p^2 - 4r)x^2 - q^2 = 0$,

ſince this Equation has been manifeſtly deduced from the Equation $-\frac{1}{2}xx-\frac{1}{2}p\mp\frac{q}{2x}=-\frac{2r}{x^2+p\mp\frac{q}{x}}$.

CXLV.

The firſt, $z=\pm\frac{1}{2}x\pm\sqrt{-\frac{1}{4}xx-\frac{1}{2}p-\frac{q}{2x}}$ of the two foregoing Expreſſions is to be choſen, as being the moſt Commodious.

CXLVI.

Since the Equation from whence is deduced the Value of $x$, gives neceſſarily three Values of $x$ preceded by the Sign $\pm$, and there is no Reaſon of preferring one of thoſe Values to another, beſides, it is known, that an Equation of the fourth Degree cannot have more than four Roots, it ſeems natural to conclude, that any one of thoſe three Values of $x$ preceded by the Sign $\pm$, may be employed indifferently, and ſtill the ſame Expreſſion for the four Values of $z$ be deduced.

Let which ever of the Roots of the reduced Equation be employed the ſame Roots will always reſult for the propoſed Equation.

To render this manifeſt, the moſt natural Method would be to find the three Values of $x$ preceded by $\pm$ given by the Equation $x^6+2px^4+(pp-4r)x^2-q^2=0$, and ſubſtitute them one after the other in the Expreſſion $z=\pm\frac{1}{2}x\pm\sqrt{-\frac{1}{4}xx-\frac{1}{2}p\mp\frac{q}{2x}}$; to be aſſured of the Identity of the three different Expreſſions ariſing from thoſe Subſtitutions. But the Calculation that this Method would require is ſo long, that one could ſcarce have Patience to continue it to the End, we ſhall therefore proceed to explain how the Analyſts have remedied this Inconveniency.

I obſerve firſt, that whatever Value of $x$ is ſubſtituted, in the general Expreſſion $z=\pm\frac{1}{2}x\pm\sqrt{-\frac{1}{4}xx-\frac{1}{2}p\mp\frac{q}{2x}}$, the four Values of $z$ expreſſed at once by this Quantity, may be repreſented by $i+k$, $i-k$; $-i+l$, $-i-l$; or what amounts to the ſame Thing, that the four Roots of the Equation $z^4+pzz+qz+r=0$, may be repreſented by $z-i-k$, $z-i+k$, $z+i-l$, $z+i+l$; $i$ denoting the Part $\frac{1}{2}x$ of the Value of $z$, $k$ and $l$ the two Quantities $\sqrt{-\frac{1}{4}xx-\frac{1}{2}p-\frac{q}{2x}}$, and $\sqrt{-\frac{1}{4}xx-\frac{1}{2}p+\frac{q}{2x}}$, multiplying therefore thoſe four Roots into one another, there reſults the Equation

$z^4 - (2ii + kk + ll)z^2 - (2ikk - 2ill)z + i^4 - i^2k^2 - i^2l^2 + k^2l^2 = 0$
which compared with $z^4 + pz^2 + qz + r = 0$, gives the Equations
$p = -2i^2 - k^2 - l^2$, $q = -2ikk + 2ill$, $r = i^4 - i^2k^2 - i^2l^2 + k^2l^2$,
by Means of which, the Equation $x^6 + 2px^4 + (p^2 - 4r)x^2 - qq = 0$,
is transformed into.
$x^6 - (4i^2 + 2k^2 + 2l^2)x^4 + (8i^2k^2 + 8i^2l^2 + k^4 - 2k^2l^2 + l^4)x^2 - 4i^2k^4 + 8i^2k^2l^2 - 4i^2l^4 = 0$
whose Roots are, $x = \pm 2i$; $x = \pm k \pm l$; $x = \pm k \mp l$; now, if we substitute any one of those Values of $x$ in the four Expressions contained in
$z = \pm\frac{1}{2}x \pm \sqrt{\left(-\frac{1}{4}xx - \frac{1}{2}p \pm \frac{q}{2x}\right)}$, or rather in
$z = \pm\frac{1}{2}x \pm \sqrt{\left(-\frac{1}{4}xx + ii + \frac{1}{2}k^2 + \frac{1}{2}l^2 \mp \frac{il^2 - ik^2}{x}\right)}$, & there will still result the four Values of $z$, $i + k$, $i - k$, $-i + l$, $-i - l$.

CXLVII.

It follows from hence, that the Roots of an Equation of the fourth Degree are all four real, or all four imaginary; or of the four Roots two are real and the other two imaginary, for it is not possible to make any other Suppositions with Respect to the four Roots of the given Equation, since it has been proved that those four Roots are all expressed at once by the Formula $z = \pm\frac{1}{2}x \pm \sqrt{\left(-\frac{1}{4}xx - \frac{1}{2}p \mp \frac{q}{2x}\right)}$.

An Equation of the fourth degree is exactly resolvable when of its four roots two are real and two imaginary.

It follows also from hence, that when of the four Roots of an Equation of the fourth Degree two are imaginary, and two are real, the reduced $x^6 + 2px^4 + (p^2 - 4r)x^2 - qq = 0$, will be of the Number of Equations solvable by the Formula of Art. CXXVI, and consequently in this Case the Equation $z^4 + pz^2 + qz + r = 0$, can be compleatly solved. but the contrary happens when the four Roots are all real or all imaginary. That this may appear, we are to consider that the general reduced Equation
$x^6 - (4i^2 + 2l^2 + 2k^2)x^4 + (8i^2k^2 + 8i^2l^2 + k^4 - 2k^2l^2 + l^4)x^2 - 4i^2k^4 + 8i^2k^2l^2 - 4i^2l^4 = 0$
is the Product of three Roots $xx - 4ii$, $xx - k^2 - 2kl - ll$, $x^2 - k^2 + 2kl - ll$:
Now, if one of the two Quantities $k$ or $l$ only are imaginary, the two Roots, $x^2 - kk - 2kl - ll$, $xx - kk + 2kl - ll$ are imaginary, and consequently the reduced Equation is in this Case solvable by the Formula of Art. CXXVI.

The contrary happens when the four roots are all real or all imaginary.

But if $k$ and $l$ are both real, or both imaginary, the three Roots $xx - 4ii$, $xx - kk - 2kl - ll$, $xx - kk + 2kl - ll$ of the reduced Equation will be all three real, and consequently cannot be obtained by the Formula of Art. CXXVI. whence in the fourth Degree as in the Third the Formulas of the Resolution can be applied only to Equations that have two impossible Roots.

CXLVIII.

How to distinguish the case of the four real roots from that of the four imaginary ones.

When an Equation of the fourth Degree is proposed to be solved, and by its Means the reduced Equation $x^6 - 2p x^4 + (pp - 4r) x^2 - qq = 0$, has been formed, if it be found to be of the Number of those that are not resolvable by the Formula of Art. CXXVI. and it is proposed to find whether in this Case the four Roots are real, or whether they are all four imaginary, it may be effected from the two following Observations.

Conditions of the four real roots.

1°. When the four Roots are real, the general reduced Equation $x^6 - 2\times(2i^2+k^2+l^2)x^4 + [8i^2(k^2+l^2)+(k^2-l^2)^2]x^2 - 4i^2(k^2-l^2)^2 = 0$, has necessarily the second Term negative, and the third Term positive, since $-2\times(2i^2+k^2+l^2)$ can neither vanish, nor be positive, when $i$, $k$, $l$, are real Quantities, nor can $8ii\times(k^2+l^2)+(k^2-l^2)^2$ likewise vanish, or be negative, under the same Restrictions.

Conditions of the four imaginary roots.

2°. But if the four Roots are imaginary, or what amounts to the same Thing, if $kk$ and $ll$ are negative, the reduced which is then expressed thus, $x^6 + 2(k^2+l^2-2ii)x^4 + [(k^2+l^2)(k^2+l^2-8ii) - 4k^2l^2]x^2 - 4i^2(k^2-l^2)^2 = 0$, cannot have at the same Time the second Term negative, and the third positive, for if $k^2+l^2$ is less than $2i^2$ which would render the second Term negative, the third Term whose Coeficient is $(k^2+l^2)\times(k^2+l^2-8i^2)-4k^2l^2$ will be necessarily negative.

CLXIX.

Every Equation of the fourth degree wanting the second term and whose third term is positive has imaginary roots.

The Inspection of the Equation $z^4-(2i^2+k^2+l^2)z^2-(2ik^2-2ill)z$—&c. given in the Art. CXLVI. furnishes an Observation whereby sometimes may be discovered, if an Equation that should have its four Roots either all real or all imaginary, is in the first of those two Cases or in the second. The Observation is, that every Equation of the fourth Degree, wanting the second Term, having the third Term positive, necessarily contains impossible Roots. Since the third Term of all those Equations represented by $(-2ii+kk+ll)zz$ can never be positive, when $ii$, $kk$, $ll$, are positive; that is, when the Roots are real. Knowing therefore that an Equation of the fourth Degree has impossible Roots, and being assured on the other Hand, that it should have its four Roots all real, or all impossible, it is easy to perceive which of those two Cases should take Place.

CL.

How to obtain approximate values

When it has been found, that the four Roots of an Equation of the fourth Degree are all real, one of the Roots of its reduced Equation is to be sought by the Method of Art. CXXIX. and being substituted in the general Formula

$z = \pm \frac{1}{2} x \pm \sqrt{\left[ -\frac{1}{4} x x - \frac{1}{2} p \mp \frac{q}{2x} \right]}$, there will result the Values of the four Roots required.

of the four roots when they are real.

CLI.

To apply the foregoing Rules to the Solution of the Equation, $u^4 - au^3 + \frac{a^2-b}{2} \times u^2 - \frac{a^3 - 3ab + 2c}{6} \times u + \frac{a^4 - 6a^2b + 8ac + 3b^2 - 6d}{24} = 0$, produced by the foregoing Problem, let $a = -16$, $b = 58$, $c = -28$, $d = -2222$, whereby the Equation will be reduced to $u^4 + 16u^3 + 99u^2 + 228u + 144 = 0$, and substituting in this Equation $z - 4$ for $u$, to make the 2d Term vanish, it will be transformed into $z^4 + 3z^2 - 52z + 48 = 0$, which being compared with the general Equation $z^4 + pz^2 + qz + r = 0$, will give $p = 3$; $q = -52$; $r = 48$; and consequently the reduced Equation will be $x^6 + 6x^4 - 183x^2 - 2704 = 0$, in which exterminating the second Term, by substituting $y - 2$ in the Room of $x^2$, there will result $y^3 - 195y - 2322 = 0$, which is resolvable by the Formula of Art. CXXVI. and shews consequently, that the proposed Equation is of the Number of those that can be exactly solved, that is, of those that have two real and two impossible Roots. In order to find them, I employ the Formula of Art. CXXVI. to solve $y^3 - 195y - 2322 = 0$, and the Value it gives is $y = \sqrt[3]{(1161 + \sqrt{1073296})} - \sqrt[3]{(-1161 + \sqrt{1073296})}$ which is reduced to $\sqrt[3]{(1161 + 1036)} - \sqrt[3]{(-1161 + 1036)}$, or to $\sqrt[3]{2197} + \sqrt[3]{135}$, or at Length to 18; substituting this Value of $y$ in $x = \sqrt{(y - 2)}$, there results $x = \sqrt{16} = 4$, which substituted in the general Value of $z = \pm \frac{1}{2} x \pm \sqrt{\left( -\frac{1}{4} x x - \frac{p}{2} \mp \frac{q}{2x} \right)}$, there will result for the two real Roots of the Equation $z^4 + 3z^2 - 52z + 48 = 0$; $z = 2 \pm \sqrt{\left( -4 - \frac{3}{2} + \frac{13}{2} \right)}$, or $z = 2 \pm 1$, that is, either 3, or 1, and for the two impossible Roots $z = -2 - \pm \sqrt{\left( -4 - \frac{3}{2} - \frac{13}{2} \right)}$, or $z = -2 \pm \sqrt{-12}$. Then substituting these four Values in $u = z - 4$, there will result for the four Roots of the proposed Equation

The Equation produced by the proposed Problem solved by the foregoing method.

$u^4+16u^3+99u^2+228u+144=0$, $u=-1$, $u=-3$, $u=-6\pm\sqrt{-12}$.

CLII.

*The Sum* (a), *and the Sum of the Cubes* (b), *of five Numbers in continued geometrical Proportion being given; to find the Numbers.*

Let $x$, $z$ and $y$ denote the three middle Numbers taken in Order. Then $\frac{x^2}{z}$ will be the first Number, and $\frac{y^2}{z}$ the last; and we shall have $\frac{x^2}{z}+x+z+y+\frac{y^2}{z}=a$, $\frac{x^6}{z^3}+x^3+z^3+y^3+\frac{y^6}{z^3}=b$ by the Question. Put $u=x+y$, then from the first Equation $\frac{x^2}{z}+\frac{y^2}{z}=a-u-z$, and cubing the two Members $\frac{x^6}{z^3}+\left[\frac{3x^2}{z}+\frac{3y^2}{z}\right]\times\frac{x^2y^2}{z^2}+\frac{y^6}{z^3}=(a-u-z)^3$, whence $\frac{x^6}{z^3}+\frac{y^6}{z^3}=(a-u-z)^3-(3a-3u-3z)\times z^2$, and $x^3+y^3=(x+y)^3-(x+y)\times 3xy=u^3-3uz^2$, & substituting those Values above, we have $(a-u-z)^3-(3a-3u-3z)\times z^3+u^3-3uz^2+z^3=b$ and $az-u^2-uz+z^2=0$, the first of which Equations by Reduction, becomes $a^3-3a^2\times(u+z)+3a\times(u^2+2uz)-3u^2z-3uz^2+3z^3=b$, from whence the other Equation multiplied by $3z$ being subtracted, there remains $a^3-3a^2\times(u+z)+3a\times(u^2+2uz-z^2)=b$, therefore $u^2+2uz-z^2-a\times(u+z)=\frac{b}{3a}-\frac{a^2}{3}$, but, by the second Equation, $u^2+2uz-z^2=az+uz$, whence, by Substitution $az+uz-a\times(u+z)=\frac{b}{3a}-\frac{a^2}{3}$; that is, $uz-az=\frac{b}{3a}-\frac{a^2}{3}$. or $az-uz=d$ $\left(\text{by putting } \frac{a^2}{3}-\frac{b}{3a}=d\right)$ from which the second Equation being subtracted, there results $u^2-z^2=d$, wherein let $\frac{az-d}{z}$ the Value of $u$ (found from the former Equation) be now substituted, and we shall have $\frac{(az-d)^2}{z^2}-z^2=d$, and consequently, there will result $z^4-(a^2-d)\times z^2+2adz-d^2=0$, whence $z$ will be found.

Another Problem producing an Equation of the fourth degree solved by the foregoing method.

CLIII.

Now let $a$ and $b$ be such that $-a^2+d=3$, $2ad=2$, and $d^2=3$, whereby the foregoing Equation will be transformed into $z^4+3z^2+2z-3=0$, Comparing this Equation with the general Equation $z^4+pz^2+qz+r=0$, we will have $p=3$, $q=2$, $r=-3$. And these Values being substituted in the reduced Equation $x^6+2px^4+(pp-4r)xx-qq=0$, will transform it into $x^6+6x^4+21x^2-4=0$.

To solve this Equation, I put $x^2=u-2$, to make the second Term vanish, whereby it will be reduced to $u^3+9u-30=0$, which according to Art. CXXXIV. is of the Number of those which have only one real Root, and consequently may be resolved by Means of the general Formula of Art. CXXVI. Whence it appears that the proposed Equation has two real Roots and two imaginary ones.

That the proposed Equation has imaginary Roots, might be perceived by observing that the Coefficient of its third Term $2z$ is positive, (Art. CLIII.

Resolving now the Equation $u^3+9u-30=0$, by the Formula of Art. CXXVI. I find $u=\sqrt[3]{(15+6\sqrt{7})}+\sqrt[3]{(15-6\sqrt{7})}$ wherefore $\pm\sqrt{(u-2)}$, or $x=\pm\sqrt{[\sqrt[3]{15+6\sqrt{7}}+\sqrt[3]{(15-6\sqrt{7})}-2]}$.

This Value of $x$ being substituted in the general Value, or Formula $z=\pm\frac{1}{2}x\pm\sqrt{\left(-\frac{1}{4}xx-\frac{p}{2}\mp\frac{q}{2x}\right)}$, which in the present Case is $x=\pm\frac{1}{2}x\pm\sqrt{\left(-\frac{1}{4}x^2-\frac{3}{2}\mp\frac{1}{x}\right)}$ there will result for the two real Roots, $z=-\frac{1}{2}\sqrt{[\sqrt[3]{(15+6\sqrt{7})}+\sqrt[3]{(15-6\sqrt{7}-2)}]}$

$$\pm\sqrt{\left[-\tfrac{1}{4}\sqrt[3]{(15+6\sqrt{7})}-\tfrac{1}{4}\sqrt[3]{(15-6\sqrt{7})}-1+\frac{1}{\sqrt{[\sqrt[3]{(15+6\sqrt{7})}+\sqrt[3]{(15-6\sqrt{7})}-2]}}\right]}$$

and the two imaginary Ones. $z=\frac{1}{2}\sqrt{[\sqrt[3]{(15+6\sqrt{7})}+\sqrt[3]{(15-6\sqrt{7})}-2]}$

$$\pm\sqrt{\left[-\tfrac{1}{4}\sqrt[3]{(15+6\sqrt{7})}-\tfrac{1}{4}\sqrt[3]{(15-6\sqrt{7})}-1-\frac{1}{\sqrt{[\sqrt[3]{(15+6\sqrt{7})}+\sqrt[3]{(15-6\sqrt{7})}-2]}}\right]}$$

CLIV.

*What two Numbers are those, if from the Sum of the Square of the Greater, and six Times the Lesser, be deducted the Cube of the Lesser, the Remainder will be* 8 ; *and their Product multiplied by the Greater gives* 1.

If $x$ be the Greater & $z$ the Lesser of those Numbers, then $xx+6z-z^3=8$,

Third Problem producing an Equation of the fourth degree ſolved by The foregoing method.

or $xx = z^3 - 6z + 8$, and $x^2 z = 1$, or $xx = \frac{1}{z}$, then $z^3 - 6z + 8 = \frac{1}{z}$, or $z^4 - 6z^2 + 8z - 1 = 0$, comparing this Equation with the general One, I find $p = -6$, $q = 8$, $r = -1$, and conſequently the reduced Equation will be $x^6 - 12x^4 + 40x^2 - 64 = 0$, in which putting $x^2 = u + 4$ to make the ſecond Term vaniſh, there reſults $u^3 - 8u - 32 = 0$, which has only one real Root, expreſſed by

$$\sqrt[3]{\left(16 + \frac{80}{3\sqrt{3}}\right)} + \sqrt[3]{\left(16 - \frac{80}{3\sqrt{3}}\right)} = \frac{2\sqrt[3]{(6\sqrt{3}+10)} + 2\sqrt[3]{(6\sqrt{3}-10)}}{\sqrt{3}}$$

which by employing the Method of Art. LXXXV. is reduced to $u = \frac{2 \times (1+\sqrt{3}) - 2 \times (1-\sqrt{3})}{\sqrt{3}} = 4$, ſubſtituting this Value of $u$ in $x = \sqrt{(u+4)}$, there reſults $x = \sqrt{8}$, whereby the general Equation $z = \pm \frac{1}{2} x \pm \sqrt{\left(-\frac{1}{4} xx - \frac{p}{2} \mp \frac{q}{2x}\right)}$ is transformed into $z = \mp\sqrt{2} \pm \sqrt{(1 \mp \sqrt{2})}$, which gives $-\sqrt{2} \pm \sqrt{(1+\sqrt{2})}$ for the two real Roots of the propoſed Equation, and $+\sqrt{2} \pm \sqrt{(1-\sqrt{2})}$ for the two imaginary Ones.

CLV.

*Let it be required to divide the Number* 5 *into two ſuch Parts, that if the Square of the Leſſer be deducted from it, and the Remainder multiplied by the Square of the Leſſer, and four Times the Leſſer deducted from the Product, the laſt Remainder will be* 29.

Fourth problem producing an Equation of the fourth degree ſolved by the foregoing method.

Let $z =$ the leſſer Part, then $(5 - z^2) \times z^2 - 4z = 29$, or $z^4 - 5z^2 + 4z + 29 = 0$. This Equation compared with $z^4 + pz^2 + qz + r = 0$; gives $p = -5$, $q = 4$, $r = 29$, and conſequently the reduced Equation will be $x^6 - 10x^4 - 91x^2 - 16 = 0$, which by putting $x^2 = u + \frac{10}{3}$, will be transformed into $u^3 - \frac{373}{3}u - \frac{10622}{27} = 0$, Now, as this Equation cannot be ſolved by the Formula of Art. CXXVI. and conſequently is of the Number of thoſe whoſe three Roots are real, it follows that the Roots required of the Equation $z^4 - 5z^2 + 4z + 29 = 0$, are either all four real or imaginary, but obſerving that when the Roots are real, the ſecond Term of the reduced Equation is negative and the third poſitive, I conclude that the propoſed Equation has all its Roots imaginary ſince the third Term $-91x^2$ of its reduced Equation is negative.

CLVI.

*A Number of Yards of two Sorts of Cloth were bought for* 11 l. *each Yard of the firſt Sort coſt as many Pounds as there were Yards, and each Yard of the ſecond Sort coſt* 1 l. *Now, if the Number of Yards of the two Sorts of Cloth were interchanged, the Coſt (regulated as before) would have been but* 9 l. *How many Yards of each were there.*

Let the Number of Yards of each Sort of Cloth be expreſſed by $z$ & $y$ reſpectively, then $z^2 + y = 11$, or $zz = 11 - y$, and $yy + z = 9$, or $z = 9 - yy$, conſequently $81 - 18y^2 + y^4 = z^2$, whence $11 - y = y^4 - 18y^2 + 81$, or $y^4 - 18y^2 + y + 70 = 0$.

Fifth Problem producing an Equation of the fourth degree ſolved by the foregoing method.

To ſolve this Equation, I firſt find the reduced Equation $x^6 - 36x^4 + 44x^2 - 1 = 0$, or $u^3 - 388u - 2929 = 0$, to which it is reduced by putting $x^2 = u + 12$.

Now, as this Equation has its three Roots real, and that the ſecond Term $36x^4$ is negative, whilſt the third $44x^2$ is poſitive, it follows Art. CXLVIII. that the propoſed Equation has all its four Roots real;

To find thoſe Roots I employ the Method of Art. CXXXVII. to ſolve the Equation $u^3 - 388u - 2929 = 0$, and ſubſtitute 22, 74, the Value of $u$, in the Equation $x = \sqrt{(u + 12)}$, which gives 5, 894 for $x$, and ſubſtituting this Value of $x$ in $y = \pm \frac{1}{2}x - \sqrt{\left(-\frac{1}{4}x^2 - \frac{1}{2}p \mp \frac{q}{2x}\right)}$ there will reſult 3,426; 2,467; $-2,315$, $-3,579$ for the four Roots of the propoſed Equation, of which only 2,467 is to the Purpoſe.

CLVII.

Having ſhewn in the Reſolution of Equations of the ſecond, third, and fourth Degree, how the Analyſts by Means of the radical Signs have been able to expreſs the Value of the unknown Quantity in thoſe Equations, it remains to explain how the Equation from whence any propoſed radical Expreſſion is derived may be found, for Example, how to find the Equation whoſe Root is $x = \sqrt[3]{ab^3} + \sqrt[3]{a^2b} + \sqrt[3]{a^2c}$, that, in which $x = \sqrt{(a^3 + b^3)} - \sqrt[3]{(a^3 - b^3)}$, &c.

Method of clearing any propoſed Equation of Radicals.

To ſolve all Problems of this Kind, or which comes to the ſame, to take away any Number of Radicals out of an Equation, the Analyſts operate as follows, they ſubſtitute in the Room of each Radical an unknown Quantity, whence there reſults 1°. A new Equation clear of Radicals, 2°. As many Equations conſiſting of two Terms as there were Radicals in the propoſed Equation. Now each of thoſe Equations conſiſting of two Terms, will be

cleared of Radicals by raiſing its two Members to the Power indicated by the Exponent of the radical Sign that one of its two Terms contains. Whence we have no more to do than to exterminate out of all thoſe Equations cleared of Radicals, the unknown Quantities that have been introduced.

CLVIII.

Let it be propoſed to take away the Radicals out of the Equation $x = \sqrt[3]{a\,b^2} + \sqrt[3]{a\,d^2}$, by putting $\sqrt[3]{a\,b^2} = y$, and $\sqrt[3]{a\,d\,d} = z$, there reſults the three Equations $x = y + z$, $y^3 = a\,b^2$, $z^3 = a\,d^2$, deducing from the firſt $y = x - z$, and ſubſtituting it in the Second, there reſults $x^3 - 3\,x^2\,z + 3\,x\,z^2 - z^3 = ab^2$, out of which, it remains to exterminate $z$ by Means of the Equation $z^3 = a\,d^2$.

The foregoing method illuſtrated by an example.

To effect which, I ſubſtitute in the firſt of thoſe two Equations $x^3 - 3\,x^2\,z + 3\,x\,z^2 - z^3 = a\,b^2$ in the Room of $z^3$, $a\,d^2$ given by by the Second, and it becomes $x^3 - 3\,x^2\,z + 3\,x\,z^2 - a\,d^2 = a\,b^2$, whence is deduced $z^2 = \frac{a\,d^2 + a\,b^2 - x^3 + 3\,x^2\,z}{3\,x}$; multiplying afterwards the two Members of this Equation by $z$, and ſubſtituting in the Room of $z^3$ its Value $a\,d^2$, there reſults a new Equation $a\,d^2 = \frac{ad^2z + ab^2z - x^3z + 3x^2zz}{3x}$ which gives $z\,z = \frac{3\,a\,d^2\,x - a\,d^2\,z - a\,b^2\,z + x^3\,z}{3x^2}$.

Equating thoſe two Values of $zz$, I deduce an Equation in which $z$ is only of one Dimenſion, and ſolving it, I find $z = \frac{x^4 + 2\,a\,d^2\,x - a\,b^2\,x}{a\,b^2 + a\,d^2 + 2x^3}$ which being ſubſtituted in one of the foregoing Equations, for Example, in $x^3 - 3\,x^2\,z + 3\,x\,z^2 - a\,d^2 = a\,b^2$, gives at Length $x^9 - 3\,a\,d^2\,x^6 - 3\,ab^2x^6 + 3a^2b^4x^3 + 3a^2d^4x^3 - 21\,a^2d^2b^2x^3 = a^3b^6 + 3\,a^3b^4d^2 + 3a^3d^4b^2 + a^3d^6$.

CLIX.

Sometimes Equations may be cleared of Radicals without having Recourſe to the foregoing Method, by only tranſpoſing the Terms, and raiſing the two Members to the Power indicated by the Radical which then will be alone in one of the Members, for Example, if it was propoſed to clear the Equation $x = y + \sqrt[3]{(a^3 + \sqrt{a^5\,x})}$ of Radicals, tranſpoſing $y$ into the firſt Member, $x - y = \sqrt[3]{(a^3 + \sqrt{a^5\,x})}$, and raiſing both Members to the Cube, $x^3 - 3\,x^2\,y + 3\,x\,y^2 - y^3 = a^3 + \sqrt{a^5\,x}$, and tranſpoſing $a^3$ we have $x^3 - 3\,x^2\,y + 3\,x\,y^2 - y^3 - a^3 = \sqrt{a^4\,x}$, and ſquaring both Members there will reſult an Equation clear of Radicals.

## CHAP. IV.

*Of the Nature and the Number of the Roots of Equations of all Degrees; of the Method of finding the Equations of a lower Degree that are their Diviſors, with the Methods of approximating to the Roots of both numeral and literal Equations of every Degree.*

THE Reſolution of an Equation of the fifth Degree might perhaps be made to depend on the Reſolution of one of the Fourth, that of an Equation of the ſixth Degree on the Reſolution of one of the Fifth, &c. But this Method the Analyſts have not purſued.

Having found that Equations of the ſecond Degree have two Roots, thoſe of the Third three, thoſe of the Fourth four; they were inclined to think that in general an Equation has as many Roots as there are Units in the Number of its Dimenſions, and to be aſſured of this Truth, inſtead of inveſtigating the Roots of an Equation, they ſought the Equation that would have given Quantities for its Roots, after the Manner we are about to explain in the Reſolution of the following Problem.

I.

*The Sum (a), and the Sum of the m Powers (c) of four Numbers in continued geometrical Proportion being given, to determine the Numbers.*

Problem producing an Equation the moſt complicated of the mth degree.

If $u$ and $y$ be aſſumed to repreſent the two middle Numbers, then from the Nature of continued Proportionals, the two Extreams will be expreſſed by $\frac{uu}{y}$ and $\frac{yy}{u}$, and ſo we ſhall have $\frac{u^2}{y} + u + y + \frac{y^2}{u} = a$, and $\frac{u^{2m}}{y^m} + u^m + y^m + \frac{y^{2m}}{z^m} = b$, now putting $u + y = x$, and $uy = z$, we will have $u^m + y^m = x^m - mx^{m-2}z + m \times \frac{m-3}{2} x^{m-4} z^2 -$

$m \times \frac{m-4}{2} \times \frac{m-5}{3} x^{m-6} z^3 + m \times \frac{m-5}{2} \times \frac{m-6}{3} \times \frac{m-7}{4} x^{m-8} z^4 -$ &c.

and becauſe $\frac{u^2}{y} + \frac{y^2}{u} = a - x$, the Sum of the *mth* Powers of the two Extreams $= (a-x)^m - m \times (a-x)^{m-2} z + m \times \frac{m-3}{2} (a-x)^{m-4} z^2$ &c.

we therefore have

$x^m + (a-x)^m - mz \times [x^{m-2} + (a-x)^{m-2}] + \frac{m}{1} \times \frac{m-3}{2} z^2 \times [x^{m-4} + (a-x)^{m-4}]$ &c.

which Equation by writing $\frac{x^3}{2x+a}$ inſtead of its equal $z$, becomes

$x^m + (a-x)^m - \frac{mx^3}{2x+a} \times [x^{m-2} + (a-x)^{m-2}] + \frac{m(m-3)}{2} \times \frac{x^6}{(2x+a)^2}$

$\times [x^{m-4} + (a-x)^{m-4}$ &c.] which being ſolved, the Value of $x$ will be found, from which the ſeveral Values of $z$, $u$, $y$ will alſo become known.

II.

Let $x^m + px^{m-1} + qx^{m-2}$ + &c. - - - - - - - $r = 0$, expreſs in general an Equation of the $m$th Degree clear of Radicals and Fractions. To find the Value of $x$ in this Equation, is the ſame as to find a Quantity, poſitive, or negative, real or imaginary, which ſubſtituted in $x^m + px^{m-1} + qx^{m-2}$ + &c. - - - $r = 0$, in the Place of $x$, will make all the Terms vaniſh. Let us ſuppoſe this Quantity $a$ to be found, I ſay that $x^m + px^{m-1} + qx^{m-2}$ + &c. - - - $r = 0$, will be diviſible by $x-a$ for it is manifeſt 1°. that ſince $x$ is only of one Dimenſion in the Diviſor, by the Rules of Diviſion the Operation may be carried on until a Remainder is obtained that does not contain $x$. Let then $Q$ be the Quotient, it is manifeſt, that if to the Product of the Quotient $Q$ into the Diviſor $x-a$, the Remainder $R$ be added, there will reſult a Quantity equal to the Dividend, now by ſubſtituting in the Dividend, $a$ for $x$, all the Terms by Hypotheſis vaniſh, therefore the Quantity $(x-a) \times Q + R$ ſhould alſo vaniſh, if $x$ be put $= a$, but by putting $x = a$, this Quantity becomes $(a-a)Q + R$, and ſince $(a-a)Q + R = 0$, $R = 0$, wherefore the Diviſion is performed without leaving a Remainder, wherefore $x^m + px^{m-1} + qx^{m-2}$ + &c. - - - $r$ is exactly diviſible by $x-a$.

An Equation may be reſolved into as many ſimple factors as there are units in the higheſt dimenſion of the unknown quantity.

Let a Quantity $b$ when ſubſtituted in the Room of $x$ in the Quotient $Q$ reſulting from the Diviſion, make all the Terms of this Quotient vaniſh, it will be diviſible by $x-b$; and it is manifeſt that if $b$ when ſubſtituted in the Room of $x$ makes all the Terms of the Quotient $Q$ vaniſh, it will alſo make the Dividend vaniſh, for the Dividend, $= (x-a)Q$ therefore every Suppoſition that reduces $Q$ to nothing, will alſo reduce the Dividend to nothing, wherefore $x-b$ will alſo divide the Dividend exactly.

In like Manner, if $c$ be a Quantity which ſubſtituted in the Room of $x$, will make the Quotient $Q$ divided by $x-b$ vaniſh, the new Quotient and conſequently the Dividend will be diviſible by $x-c$, we will therefore have as many ſimple Quantities $x-a$, $x-b$, $x-c$, &c. as there are Units in $m$, which multiplied by one another, will produce the propoſed Equation, in the Room, therefore of the propoſed Equation we may ſubſtitute $(x-a)(x-b)(x-c)$ &c. $= 0$.

III.

When a Quantity is ſuppoſed equal to nothing, one of its Factors muſt be equal to nothing, whence the propoſed Equation is the Product of $x - a = 0$ into $(x - b)$ $(x - c)$ &c. or of $x - b = 0$ into $(x - a)$ $(x - c)$ &c. in each of thoſe Caſes $x$ is the ſame Quantity in the ſame Caſe, and different in the different Caſes, thus $x^2 - (a+b)x + ab$ is the Product of $x - a = 0$ into $(x - b)$ or of $x - b = 0$ into $x - a$ this Equation $x^2 - (a + b)x + ab$ repreſents theſe two $aa - aa - ab + ab = 0$ that reſults by ſubſtituting $a$ for $x$, and $bb - ab - bb + ab = 0$ that reſults by ſubſtituting $b$ for $x$. In the firſt of theſe Caſes, $x$ and its Powers, repreſent $a$ and its Powers. in the ſecond, the Letter $x$ and its Powers repreſent $b$ and its Powers, whence the Equation $x^m + px^{m-1} + qx^{m-2} - - - - - r = 0$ really repreſents as many ſimple Equations as there are Units in the higheſt Dimenſion $m$ of the unknown Quantity.

An Equation admits of as many ſolutions as there are ſimple factors multiplied by one another that produce it.

IV.

The Product of $x - a$ into $x - b$ cannot be equal to another Product $x - e$ into $x - f$ for if it was then $\frac{x-a}{x-f} = \frac{x-e}{x-b}$ and conſequently $x - a$ muſt be diviſible by $x - f$ as alſo $x - e$ by $x - b$, which cannot be, or $x - f$ and $x - b$ muſt have a common Diviſor, as alſo $x - a$ and $x - e$ which likewiſe is impoſſible, wherefore every Quantity $x^2 + px + q$ in which $x$ riſes to two Dimenſions cannot be produced but by the Multiplication of two ſimple Factors $x - a$, $x - b$, and by no other than thoſe two, wherefore in an Equation of the ſecond Degree $x$ cannot poſſibly have more than two different Values $a$ and $b$.

No Equation can have more roots than it contains dimenſions of the unknown quantity.

In like Manner it will appear, that the Product of $x - a$ into $x - b$ into $x - c$ cannot be equal to $x - e$ into $x - f$ into $x - g$, for then $\frac{x-a}{(x-f)(x-g)} = \frac{x-e}{(x-b)(x-c)}$, wherefore the Denominators of thoſe Factors ſhould have a common Diviſor as likewiſe their Numerators $x - a$, $x - e$, which cannot be, wherefore in an Equation of the third Degree, and in general in any Equation, the unknown Quantity cannot have more Values either real or imaginary than there are Units in the Degree of the Equation.

V.

Objection against the foregoing demonstration solved.

Against the foregoing Demonstration the following Objection may be started. Let $a = 4$, $b = 17$, $c = 7$, $e = 8$ and $x = 2$, then $(x - a)(x - b) = -2 \times -15 = -5 \times -6 = (x - 7)(x - 8) = (x - c)(x - e)$. it is true, that in some Cases by giving $x$ a certain Value $(x - a)(x - b) = (x - c)(x - e)$, but as $x$ is a general and undetermined Quantity, the foregoing Equation should take Place, whatever Value is given to $x$ which is impossible, in Effect let $x = a$ then $(a - a)(a - b) = (a - c)(a - e)$ that is, $(a - c)(a - e) = 0$, which cannot be, since $c$ and $e$ differ from $a$ and $b$.

VI.

General form that may be given to every imaginary quantity whatsoever.

Some of those Quantities $a$, $b$, $c$, &c. or all of them, may denote real Quantities, equal, or unequal, or simple imaginary Quantities, as $B\sqrt{-1}$, or mixed ones, as $A + B\sqrt{-1}$, to those two Forms all imaginary Quantities whatsoever being reducible, $A$ and $B$ expressing real Quantities.

1°. $a + b\sqrt{-1} \pm g + b\sqrt{-1}$, may be reduced to the Form of $A + B\sqrt{-1}$, for $a + b\sqrt{-1} \pm g \pm b\sqrt{-1} = a \pm g + (b \pm b)\sqrt{-1} = A + B\sqrt{-1}$, by putting $a \pm g = A$, and $b \pm b = B$.

2°. $(a + b\sqrt{-1}) \times (g + b\sqrt{-1})$ may be reduced to $A + B\sqrt{-1}$. for $(a + b\sqrt{-1}) \times (g + b\sqrt{-1}) = (ag - bb + bg + ab)\sqrt{-1}$, wherefore $ag - bb = A$, and $bg + ab = B$.

3°. $\frac{a + b\sqrt{-1}}{g + b\sqrt{-1}} = A + B\sqrt{-1}$, for $\frac{a + b\sqrt{-1}}{g + b\sqrt{-1}} = \frac{(a + b\sqrt{-1}) \times (g - b\sqrt{-1})}{(g + b\sqrt{-1}) \times (g - b\sqrt{-1})} = \frac{(ag + bb + bg - ab)\sqrt{-1}}{gg + bb}$, wherefore $\frac{ag + bb}{gg + bb} = A$, and $\frac{bg - ab}{gg + bb} = B$.

4°. $(a + b\sqrt{-1})^{g + b\sqrt{-1}} = A + B\sqrt{-1}$, for supposing it to be so, $Log.\ (a + b\sqrt{-1})^{g + b\sqrt{-1}} = Log.\ (A + B\sqrt{-1})$, or $(g + b\sqrt{-1})\ Log.\ (a + b\sqrt{-1}) = Log.\ (A + B\sqrt{-1})$, wherefore since the limiting Ratio of the cotemporary Increments of $A + B\sqrt{-1}$, and of its *Logaritbm*, is equal to the limiting Ratio of the

cotemporary Increments of $(a+b)^{g+b\sqrt{-1}}$ and of its *Logarithm*, $g+b\sqrt{-1}$ being ſuppoſed to be a permanent Quantity, we will have $(N)\ (g+b\sqrt{-1})\times\left(\frac{da+db\sqrt{-1}}{a+b\sqrt{-1}}\right)=\frac{dA+dB\sqrt{-1}}{A+B\sqrt{-1}}$, and

$$(g+b\sqrt{-1})\times\left(\frac{da+db\sqrt{-1}}{a+b\sqrt{-1}}\right)=(g+b\sqrt{-1})\times\frac{(da+db\sqrt{-1})(a-b\sqrt{-1})}{(a+b\sqrt{-1}(a-b\sqrt{-1})}$$

$$=\frac{gada+gbdb-abdb+bbda}{aa+bb}+\frac{(bada+bbdb+gadb-gbda)\sqrt{-1}}{aa+bb}$$

and performing the ſame Operation upon the ſecond Member of the Equation $(N)$, we will have $\frac{dA+dB\sqrt{-1}}{A+B\sqrt{-1}}=\frac{(dA+dB\sqrt{-1})\times(A-B\sqrt{-1})}{(A+B\sqrt{-1})\times(A-B\sqrt{-1}}$

$=\frac{AdA+BdB}{AA+BB}\times\frac{(AdB-BdA)\sqrt{-1}}{AA+BB}$, and conſequently

$$\frac{gada+gbdb-abdb+bbda}{aa+bb}+\frac{(bada+bbdb+gadb-gbda)\sqrt{-1}}{aa+bb}$$

$=\frac{AdA+BdB+(AdB-BdA)\sqrt{-1}}{AA+BB}$. Now becauſe the real Part of the firſt Member is equal to the real Part of the Second, and the imaginary Part to the imaginary Part, we will have the two following Equations, $(O)$ $\frac{gada+gbdb-abdb+bbda}{aa+bb}=\frac{AdA+BdB}{AA+BB}$

& $(P)$ $\frac{(bada+bbdb+gadb-gbda)\sqrt{-1}}{aa+bb}=\frac{(AdB-BdA)\sqrt{-1}}{AA+BB}$

wherefore $(Q)$ $\frac{AdA+BdB}{AA+BB}=g\times\left\{\frac{ada+bdb}{aa+bb}\right\}-b\times\left\{\frac{adb-bda}{aa+bb}\right\}$

I now obſerve, that if the *Logarithm* of $\sqrt{(AA+BB)}$ be expreſſed by $x$ the limiting Ratio of the cotemporary Increments of $\sqrt{(AA+BB)}$ & of its *Logarithm* $x$ will be expreſſed by the Equation $\frac{AdA+BdB}{AA+BB}=dx$.

In like Manner, if the *Logarithm* of $\sqrt{(aa+bb)}^{g}$ be expreſſed by $y$, the Equation $g\times\frac{ada+bdb}{aa+bb}=dy$ will expreſs the limiting Ratio of the cotemporary Increments of $\sqrt{(aa+bb)}^{g}$ and of its *Logarithm* $y$.

and if we put $-b \times \left\{ \frac{a\,db - b\,da}{aa+bb} \right\} = d\,z$, and consider this Equation as expreſſing the limiting Ratio of the cotemporary Increments of two variable Quantities, and $\int -b \times \left\{ \frac{adb-bda}{aa+bb} \right\} = z$ that from which it has been deduced, or $Log.\ c^{\int -b \left\{ \frac{a\,db - b\,da}{aa+bd} \right\}} = z$, $c$ expreſſing the Baſe of the *Logarithms* Art. XLVIII. Chap. III. we will have

$$Log. \sqrt{(AA+BB)} = Log. (aa+bb)^{\frac{-b}{2}} \int \left\{ \frac{adb-bda}{aa+bb} \right\} + Log.\ c$$

$$\text{or } \sqrt{(AA+BB)} = \sqrt{(aa+bb)}^{-b \int \left\{ \frac{a\,db - b\,da}{aa+bb} \right\}} \times c$$

Performing upon the Equation ($P$) the ſame Operation that was performed upon the Equation ($O$), this Equation ($P$) after dividing both Members by $\sqrt{-1}$, and reducing becomes

$$(R)\ \frac{A\,dB - B\,dA}{AA+BB} = b \times \left\{ \frac{a\,da + b\,db}{aa+bb} \right\} + g \times \left\{ \frac{adb-bda}{aa+bb} \right\}$$

Now, $\int . b \times \left\{ \frac{a\,da + b\,db}{aa+bb} \right\} = b\ Log. \sqrt{(aa+bb)}$, as to $\int \frac{A\,dB - B\,dA}{AA+BB}$ it Expreſſes an Arc whoſe Tangent is $\frac{B}{A}$, for let $\omega$ expreſs an Arc whoſe Tangent is $\frac{B}{A}$, and let this Arc $\omega$ receive the Increment $d\omega$, then $d$ Tang. $\omega$ = Tang. $(\omega + d\omega)$ — Tang. $\omega$, but Tang. $(\omega + d\omega) = \frac{Tang.\ \omega + tang\ d\omega}{1 - tang.\ \omega \times tang.\ d\omega}$, conſequently $d\ tang.\ \omega = \frac{tang.\ d\omega + tang.^2\ \omega \times tang.\ d\omega}{1 - tang.\ \omega .\ tang\ d\omega}$ and $\frac{tang.\ d\omega}{d\ tang.\ \omega} = \frac{1 - tang.\ \omega \cdot tang.\ d\omega}{1 + tang.^2\ \omega}$ which Ratio is leſs than $\frac{1}{1 + tang.^2 \omega}$, but continually approaches to it when $d\omega$ decreaſes, and coincides with it when $d\omega$ vaniſhes, and becauſe in this Caſe $d\omega$ and $tang.\ d\omega$ continually approach to a Ratio of Equality,

$da = \frac{d.\,tang.\,a}{1 + tang.^2\,a}$ will expreſs the limiting Ratio of the cotemporary Increments of the Arc and its Tangent, in which Expreſſion ſubſtituting for $tang.\,a$ its Value $\frac{B}{A}$, and for $d.\,tang.\,a$ its Value $\frac{A\,dB - B\,dA}{A\,A}$, there reſults $d\,a = \frac{A\,dB - B\,dA}{A\,A + B\,B}$. In like Manner it will appear that $\int g \times \frac{a\,db - b\,da}{a\,a + b\,b}$ expreſſes the Product ariſing from the Multiplication of $g$ into an Arc whoſe Tangent is $\frac{b}{a}$.

Whence to reduce $(a + b\sqrt{-1})^{\frac{g}{c}}$ to the Form $A + B\sqrt{-1}$, it ſuffices to deſcribe a Circle with a Radius equal to the Value found for $\sqrt{(A\,A + B\,B)}$, that is, equal to $\sqrt{(a\,a + b\,b)}^{\frac{g}{c}}$ and take on the Circumference of this Circle, an Arc equal to the Value alſo found of $\int \left\{ \frac{A\,dB - B\,dA}{A\,A + B\,B} \right\}$ that is, equal to $h\,Log.\sqrt{(aa + bb)} + g \times \int \left\{ \frac{a\,db - b\,da}{a\,a + b\,b} \right\}$; $\frac{B}{A}$ will be the Tangent of this Arc; $B$ will be the Sinus and $A$ the Coſinus.

$$-b \int \left\{ \frac{a\,db - b\,da}{a\,a + b\,b} \right\}.$$

VII.

If one of the Terms of a Multinomial is an imaginary Radical, as for Example, $x - a - b\sqrt{-1}$, the Radical may be made to vaniſh, by multiplying it by another Multinomial that differs from it only by the Sign that precedes the Radical. Whence it is only the Product of $x - a - b\sqrt{-1}$ into $x - a + b\sqrt{-1}$, that can make the Radical in the propoſed Quantity diſappear, by giving the Product $x^2 - 2\,a\,x + a^2 + b$. Becauſe it is only in this Caſe that the particular Products of each real Term into $b\sqrt{-1}$ deſtroy each other by contrary Signs, and in the ſame Caſe it is manifeſt that the Term $b$ that contains the Product of the two Radicals $b\sqrt{-1}$ and $-b\sqrt{-1}$ is neceſſarily poſitive. Whence there are never in any Equation whoſe Coeficients are real Quantities, ſingle imaginary Roots, or an odd Number of imaginary Roots; but the Roots become imaginary in Pairs, and an Equation of an odd Number of Dimenſions, has always one real Root.

The imaginary Expreſſions cannot diſappear in the Equation produced but when their Number is even.

VIII.

Let $a$, $b$, $c$, $d$, $e$, $f$, $g$, &c. expreſs real and poſitive Quantities, $a$ being greater than $b$, $b$ greater than $c$, $c$ greater than $d$, $d$ greater than $e$, &c. The Syſtems of Factors of the Formula's of the ſecond Degree.

Method of finding the Syſtems of factors correſponding to the Formulas of the ſecond, third, &c. degree.

I. $x^2+mx+n=0$, II. $x^2-mx+n=0$, III. $x^2+mx-n=0$, IV. $x^2-mx-n$, V. $x^2+n=0$, VI. $x^2-n=0$.

In which $m$ and $n$ denote real and poſitive Numbers, will be as follows, $(x+a)(x+a)$; $(x+a)(x-a)$; $(x-a)(x-a)$; $(x+a)(x+b)$; $(x+a)(x-b$; $(x-a)(x+b)$; $(x-a)(x-b)$; $(x+a\sqrt{-1})(x-a\sqrt{-1})$; $(x+a+a\sqrt{-1})(x+a-a\sqrt{-1})$; $(x-a+a\sqrt{-1})(x-a-a\sqrt{-1})$; $(x+a+b\sqrt{-1})(x+a-b\sqrt{-1})$; $(x-a+b\sqrt{-1})(x-a-b\sqrt{-1})$; $(x+b+a\sqrt{-1})(x+b-a\sqrt{-1})$; $(x-b+a\sqrt{-1})(x-b-a\sqrt{-1})$;

In like Manner the Syſtems of Factors of the Formula's of the third, fourth, fifth, &c. Degree will be found.

IX.

To diſcover the Nature, that is, the Sign of each Coeficient of each Syſtem of Factors correſponding to Equations of the ſecond, third, fourth, &c. Degree, the Factors of thoſe Syſtems are to be multiplied into one another.

When ſeveral ſyſtems of Factors correſpond to the ſame Formula how to find the ſyſtem of factors correſponding to any propoſed Equation contained in this Formula.

When ſeveral Syſtems of Factors produce the ſame Formula, to determine the nature of the Roots of any propoſed Equation contained in this Formula, it will be requiſite to find the Syſtem of Factors that will produce it, how this may be effected by Means of the Coeficients of its Terms, we ſhall proceed to explain.

I obſerve that there are as many Quantities $a$, $b$, $c$, $d$, in each Syſtem of Factors, as there are Coeficients $m$, $n$, $p$, &c. Whence there will reſult as many Equations between $a$, $b$, $c$, $d$, &c. as there are Coeficients. Conſequently if ſome of the Quantities $a$, $b$, $c$, &c. are equal, there will be more Equations than unknown Quantities, by which Means an Equation will be obtained, out of which the unknown Quantities $a$, $b$, $c$, &c. have been exterminated, which will ſerve to determine whether the propoſed Equation has equal Roots or not, and in Caſe it has, the Form and Signs of its Roots.

But if all the Quantities $a$, $b$, $c$, $d$, &c. are unequal, there will reſult only as many Equations as there are unknown Quantities, and conſequently they cannot all be exterminated. But by putting ſome of them equal, a Quantity greater than 0 in ſome Caſes, and leſs than 0 in other Caſes will be obtained, which will ſerve to determine whether the Roots of any propoſed Equation are all unequal, and in Caſe they are, the form and Signs of thoſe Roots.

X.

Thus, 1°. let $x^2+mx+n=(x+a)(x+a)=x^2+2ax+a^2=0$; then $m=2a$, $n=a^2$, consequently $m^2-4n=0$.

2°. Let $x^2+mx+n=(x+a+a\sqrt{-1})(x+a-a\sqrt{-1})=x^2+2ax+2a^2=0$, then $m=2a$, $n=2a^2$. Consequently $m^2-2n=0$,

3°. Let $x^2+mx+n=(x+a)(x+b)$, if $b$ be put equal to $a$ the Quantity $m^2-4n=0$, wherefore if $b$ be leſs then $a$, the Quantity $m^2-4n$ will be greater or leſs than 0, to find which of the two ſhould take Place, let $a=2$ and $b=1$, then $m=3$, $n=2$, which Values being ſubſtituted in the Quantity $m^2-4n$ there will reſult $3^2-2^2.2>0$. Whence I conclude that $m^2-4n>0$ is the Condition for this Syſtem.

4°. Let $x^2+mx+n=(x+a+b\sqrt{-1})(x+a-b\sqrt{-1})$ if $b$ be put equal 0, this Syſtem will coincide with the firſt Syſtem, and conſequently $m^2-4n=0$, but if $b$ be greater or leſs than 0, this Quantity will be greater or leſs than 0, and on Examination it will be found to be leſs than 0, whence the Condition for this Syſtem will be $m^2-4n<0$.

If $b$ be put equal $a$, this Syſtem will coincide with the Second, and conſequently $m^2-2n=0$, but when $b$ is leſs than $a$ this Quantity will be greater or leſs than 0, and on Examination it will be found to be leſs than 0, whence the Conditions of this Syſtem are $m^2-4n<0$. $m^2-2n>0$.

5°. Let $x^2+mx+n=(x+b+a\sqrt{-1})(x+b-a\sqrt{-1})$, $b$ being put equal to $a$ this Syſtem will coincide with the Second, and conſequently $m^2-2n=0$, wherefore when $b$ is leſs than $a$, this Quantity is greater or leſs than 0, it is eaſy to perceive that in order to diſtinguiſh this Syſtem from the foregoing one, that the Condition for this Syſtem ſhould be $m^2-2n<0$.

Tables whereby are diſcovered the nature and number of the roots of any propoſed Equation.

It is thus the following Tables were conſtructed, which when compleated, will ſerve to determine in any propoſed Equation, the Nature and Number of its Roots, that is, whether they are real or imaginary, equal or unequal, poſitive or negative.

| I. $\{x^2+mx+n=0\}$ | | II. $\{(x^2-mx+n=0\}$ | |
|---|---|---|---|
| $(x+a)(x+a)$ | $m^2-4n=0$ | $(x-a)(x-a$ | $m^2-4n=0$ |
| $(x+a+a\sqrt{-1})(x+a-a\sqrt{-1})$ | $m^2-2n=0$ | $(x-a+a\sqrt{-1})(x-a-a\sqrt{-1})$ | $m^2-2n=0$ |
| $(x+a)(x+b)$ | $m^2-4n>0$ | $(x-a)(x-b)$ | $m^2-4n>0$ |
| $(x+a+b\sqrt{-1})(x+a-b\sqrt{-1})$ | $\begin{cases} m^2-4n<0 \\ m^2-2n>0 \end{cases}$ | $(x-a+b\sqrt{-1})(x-a-b\sqrt{-1})$ | $\begin{cases} m^2-4n<0 \\ m^2-2n>0 \end{cases}$ |
| $(x+b+a\sqrt{-1})(x+b-a\sqrt{-1})$ | $m^2-2n<0$ | $(x-b+a\sqrt{-1})(x-b-a\sqrt{-1})$ | $m^2-2n<0$ |

III. $x^2+mx-n=(x+a)(x-b)$ IV. $x^2-mx-n=(x-a)(x+b)$

V. $x^2+n=(x+a\sqrt{-1})(x-a\sqrt{-1})$ VI. $x^2-n=(x+a)(x-a)$

## I. $\{x^3 + mx^2 + nx + p = 0\}$.

$(x+a)(x+a)(x+a)$ - - - - - - - $m^2 - 3n = 0,\ m^3 - 27p = 0$

$(x+a)(x+a\sqrt{-1})(x-a\sqrt{-1})$ - - $m^2 - n = 0,\ m^3 - p = 0$

$(x+a)(x+a+a\sqrt{-1})(x+a-a\sqrt{-1})$ - - $4m^2 - 9n = 0,\ 2m^3 - 27p = 0$

$(x+a)(x+b)(x+b)$
$(x+a)(x+a)(x+b)$ $\Big\}$ $4(m^2-3n)(n^2-3mp)-(mn-9p)^2=0$ $\begin{cases} 2m^3-9mn+27p>0 \\ 2m^3-9mn+27p<0 \end{cases}$

$(x+a)(x+b\sqrt{-1})(x-b\sqrt{-1})$ - - - -
$(x+b)(x+a\sqrt{-1})(x-a\sqrt{-1})$ - - - - $\Big\}$ $mn-p=0$ $\begin{cases} m^2-n>0 \\ m^2-n<0 \end{cases}$

$(x+a)(x+b+b\sqrt{-1})(x+b-b\sqrt{-1})$
$(x+b)(x+a+a\sqrt{-1})(x+a-a\sqrt{-1})$ $\Big\}$ $2(m^2-n)(n^2-mp)-(mn-p)^2=0$ $\begin{cases} 4m^2-9n>0 \\ 4m^2-9n<0 \end{cases}$

$(x+b)(x+a+b\sqrt{-1})(x+a-b\sqrt{-1})$
$(x+a)(x+b+a\sqrt{-1})(x+b-a\sqrt{-1})$ $\Big\}$ $100n(m^2+n)^2-(2m^3+17mn+p)^2=0$ $\begin{cases} 4m^2-9n>0 \\ 4m^2-9n<0 \\ m^2-n>0 \\ m^2-n<0 \end{cases}$
$(x+a)(x-b+a\sqrt{-1})(x-b-a\sqrt{-1})$

$(x+b)(x+b+a\sqrt{-1})(x+b-a\sqrt{-1})$ - -
$(x+a(x+a+b\sqrt{-1})(x+a-b\sqrt{-1})$ - - $\Big\}$ $2m^3-9mn+27p=0$ $\begin{cases} 4m^2-9n<0 \\ 4m^2-9n>0 \end{cases}$

$(x+a)(x+b)(x+c)$ - - - - - - $4(m^2-3n)(n^2-3mp)-(mn-9p)^2>0$

$(x+a)(x+b+c\sqrt{-1})(x+b-c\sqrt{-1})$ - - - $\begin{cases} 4(m^2-3n)(n^2-3mp)-(mn-9p)^2<0 \\ 2(m^2-n)(n^2-mp)-(mn-p)^2>0 \\ 2m^3-9mn+27p>0,\ 4m^2-9n>0 \end{cases}$

$(x+a)(x+c+b\sqrt{-1})(x+c-b\sqrt{-1})$ $\begin{cases} mn-p>0, 2(m^2-n)(n^2-mp)-(mn-p)^2<0 \\ 100n(m^2+n)^2-(2m^3+17mn+p)^2<0, m^2-n>0 \end{cases}$

$(x+a)(x-c+b\sqrt{-1})(x-c-b\sqrt{-1})$ $mn-p<0,\ 100n(m^2+n)^2-(2m^3+17mn+p)^2<0$

$(x+b)(x+a+c\sqrt{-1})(x+a-c\sqrt{-1})$ $\begin{cases} 4(m^2-3n)(n^2-3mp)-(mn-9p)^2<0 \\ 100n(m^2+n)^2-(2m^3+17mn+p)^2<0 \\ 2m^3-9mn+27p<0, 4m^2-9n>0 \end{cases}$

$(x+b)(x+c+a\sqrt{-1})(x+c-a\sqrt{-1})$ $\begin{cases} mn-p>0, 2m^3-9mn+27p>0 \\ 100n(m^2+n)^2-(2m^3+17mn+p)^2>0, 4m^2-9n<0 \end{cases}$

$(x+b)(x-c+a\sqrt{-1})(x-c-a\sqrt{-1})$ $\begin{cases} mn-p<0, 100n(m^2+n)^2-(2m^3+17mn+p)^2>0 \\ m^2-n<0 \end{cases}$

$(x+c)(x+a+b\sqrt{-1})(x+a-b\sqrt{-1})$ $\begin{cases} 100n(m^2+n)^2-(2m^3+17mn+p)^2>0 \\ 2(m^2-n)(n^2-mp)-(mn-p)^2>0 \end{cases}$

$(x+c)(x+b+a\sqrt{-1})(x+b-a\sqrt{-1})$ - - $\begin{cases} 2m^3-9mn+27p<0 \\ 2(m^2-n)(n^2-mp)-(mn-p)^2<0 \\ 4m^2-9n<0 \end{cases}$

## II. $\{x^3 + mx^2 + nx - p = 0\}$

$(x+a)(x+a)(x-b)$ - - - - - - $4(m^2-3n)(n^2+3mp)-(mn+9p)^2=0$

$(x-b)(x+a+b\sqrt{-1})(x+a-b\sqrt{-1})$
$(x-a)(x+b+a\sqrt{-1}(x+b-a\sqrt{-1})$ $\Big\}$ $100n(m^2+n)^2-(2m^3+17mn-p)^2=0$ $\begin{cases} 2m^3-p>0 \\ 2m^3-p<0 \end{cases}$

$(x-b)(x+b+a\sqrt{-1})(x+b-a\sqrt{-1})$ - - $2m^3+mn-p=0$

$(x-b)(x+a+a\sqrt{-1})(x+a-a\sqrt{-1})$ - - $2(m^2-n)(n^2+mp)-(mn+p)^2=0$

$(x+a)(a+b)(x-c$ - - - - - - - - $4(m^2-3n)(n^2+3mp)-(mn+9p)^2>0$

$(x-a)(x+c+b\sqrt{-1})(x+c-b\sqrt{-1})$ - $\begin{cases} 100n(m^2+n)^2-(2m^3+17mn-p)^2<0 \\ 2m^3-p<0 \end{cases}$

$x-b)(x+a+c\sqrt{-1})(x+a-c\sqrt{-1})$ - $\begin{cases} 4(m^2-3n)(n^2+3mp)-(mn+9p)^2<0 \\ 100n(m^2+n)^2-(2m^3+17mn-p)^2<0 \\ 2m^3-p>0 \end{cases}$

$x-b)(x+c+a\sqrt{-1})(x+c-a\sqrt{-1})$ - $\begin{cases} 100n(m^2+n)^2-(2m^3+17mn-p)^2>0 \\ 2m^3+mn-p<0 \end{cases}$

$-c)(x+a+b\sqrt{-1})(x+a-b\sqrt{-1})$ - $\begin{cases} 100n(m^2+n)^2-(2m^3+17mn-p)^2>0 \\ 2(m^2-n)(n^2+mp)-(mn+p)^2>0 \\ 2m^3-p>0 \end{cases}$

$-c)(x+b+a\sqrt{-1})(x+b-a\sqrt{-1})$ - $\begin{cases} 2m^3+mn-p>0 \\ 2(m^2-n)(n^2+mp)-(mn+p)^2<0 \end{cases}$

## III. $\{x^3+mx^2-nx+p=0\}$

$+a)(x-b)(x-b)$ - - - - - - - - $4(m^2+3n)(-3mp+n^2)-(mn+9p)^2=0$

$+a)(x-b+b\sqrt{-1})(x-b-b\sqrt{-1})$ - - $2(m^2+n)(-mp+n^2)-(mn+p)^2=0$

$+a)(x-b)(x-c)$ - - - - - - - $4(m^2+3n)(-3mp+n^2)-(mn+9p)^2>0$

$+a)(x-b+c\sqrt{-1})(x-b-c\sqrt{-1})$ - $\begin{cases} 4(m^2+3n)(-3mp+n^2)-(mn+9p)^2<0 \\ 2(m^2+n)(-mp+n^2)-(mn+p)^2>0 \end{cases}$

$+a)(x-c+b\sqrt{-1})(x-c-b\sqrt{-1})$ - - $2(m^2+n)(-mp+n^2)-(mn+p)^2<0$

## IV. $\{x^3-mx^2+nx+p=0\}$

$-a)(x-a)(x+b)$ - - - - - - - $4(m^2-3n)(n^2+3mp)-(mn+9p)^2=0$

$+b)(x-a+b\sqrt{-1})(x-a-b\sqrt{-1})$ } $100n(m^2+n)^2-(2m^3+17mn-p)^2=0$ { $2m^3-p>0$
$+a)(x-b+a\sqrt{-1})(x-b-a\sqrt{-1})$ } { $2m^3-p<0$

$+b)(x-b+a\sqrt{-1})(x-b-a\sqrt{-1})$ - $2m^3+mn-p=0$

$+b)(x-a+a\sqrt{-1})(x-a-a\sqrt{-1})$ - $2(m^2-n)(n^2+mp)-(mn+p)^2=0$

$-a)(x-b)(x+c)$ - - - - - - - $4(m^2-3n)(n^2+3mp)-(mn+9p)^2>0$

$+a)(x-c+b\sqrt{-1})(x-c-b\sqrt{-1}$ - - $\begin{cases} 100n(m^2+n)^2-(2m^3+17mn-p)^2<0 \\ 2m^3-p>0 \end{cases}$

$+b)(x-a+c\sqrt{-1})(x-a-c\sqrt{-1})$ - $\begin{cases} 4(m^2-3n)(n^2+3mp)-(mn+9p)^2<0 \\ 100n(m^2+n)^2-(2m^3+17mn-p)^2<0 \\ 2m^3-p<0 \end{cases}$

$(x+b)(x-c+a\sqrt{-1})(x-c-a\sqrt{-1})$ - $\begin{cases} 100\,n\,(m^2+n)^2-(2m^3-17mn-p)^2>0 \\ 2m^3+mn-p<0 \end{cases}$

$(x+c)(x-a+b\sqrt{-1})(x-a-b\sqrt{-1})$ - $\begin{cases} 100\,n\,(m^2+n)^2-(2m^3+17mn-p)^2>0 \\ 2(m^2-n)(n^2+mp)-(mn+p)^2>0 \\ 2m^3-p>0 \end{cases}$

$(x+c)(x-b+a\sqrt{-1})(x-b-a\sqrt{-1})$ - $\begin{cases} 2m^3+mn-p>0 \\ 2(m^2-n)(n^2+mp)-(mn+p)^2<0 \end{cases}$

## V. $\{x^3+mx^2-nx-p=0\}$

$(x+a)(x+a)(x-a)$ - - - - - - - $m^2-n=0, m^3-p=0$

$(x+a)(x+b)(x-b)$ - - - - - - - $\left.\begin{matrix} \\ \\ \end{matrix}\right\} mn-p=0 \begin{cases} m^2-n>0 \\ m^2-n<0 \end{cases}$

$(x+a)(x-a)(x+b)$ - - - - - - -

$(x-a)(x+b)(x+b)$ - - - - - - $\left.\begin{matrix} \\ \\ \end{matrix}\right\} 4(m^2+3n)(n^2+3mp)-(-mn+9p)^2=0 \begin{cases} m^2-n<0 \\ m^2-n>0 \end{cases}$

$(x+a)(x+a)(x-b)$ - - - - - -

$(x-a)(x+b+b\sqrt{-1})(x+b-b\sqrt{-1})$ - $2(m^2+n)(n^2+mp)-(-mn+p)^2=0$

$(x-a)(x+a+b\sqrt{-1})(x+a-b\sqrt{-1})$ - $2\,m^3-mn-p=0$

$(x+a)(x+b)(x-c)$ - - - - - - $\begin{cases} 4\,(m^2+3n)(n^2+3mp)-(-mn+9p)^2>0 \\ mn-p<0, m^2-n>0 \end{cases}$

$(x+a)(x-b)(x+c)$ - - - - - - - $mn-p>0$

$(x-a)(x+b)(x+c)$ - - - - - - - $\begin{cases} 4(m^2+3n)(n^2+3mp)-(-mn+9p)^2>0 \\ mn-p<0, m^2-n<0 \end{cases}$

$(x-a)(x+b+c\sqrt{-1})(x+b-c\sqrt{-1})$ - $\begin{cases} 4(m^2+3n)(n^2+3mp)-(-mn+9p)^2<0 \\ 2(m^2+n)(n^2+mp)-(-mn+p)^2>0 \\ 2m^3-mn-p<0, m^2-n<0 \end{cases}$

$(x-a)(x+c+b\sqrt{-1})(x+c-b\sqrt{-1})$ - - $2(m^2+n)(n^2+mp)-(-mn+p)^2<0$

$(x-b)(x+a+c\sqrt{-1})(x+a-c\sqrt{-1})$ - $\begin{cases} 4(m^2+3n)(n^2+3mp)-(-mn+9p)^2<0 \\ 2m^3-mn-p>0, m^2-n>0 \end{cases}$

## VI. $\{x^3-mx^2+nx-p=0\}$

$(x-a)(x-a)(x-a)$ - - - - - - $m^3-3n=0, m^3-27p=0$

$(x-a)(x+a\sqrt{-1})(x-a\sqrt{-1})$ - - $m^2-n=0, m^3-p=0$

$(x-a)(x-a+a\sqrt{-1})(x-a-a\sqrt{-1})$ - $4m^2-9n=0, 2m^3-27p=0$

$(x-a)(x-b)(x-b$ - - $\left.\begin{matrix} \\ \\ \end{matrix}\right\} 4(m^2-3n)(n^2-3mp)-(mn-9p)^2=0 \begin{cases} 2m^3-9mn+27p>0 \\ 2m^3-9mn+27p<0 \end{cases}$

$(x-a)(x-a)(x-b)$ - -

$(x-a)(x+b\sqrt{-1})(x-b\sqrt{-1})$ - $\left.\begin{matrix} \\ \\ \end{matrix}\right\} mn-p=0 \begin{cases} m^2-n>0 \\ m^2-n<0 \end{cases}$

$(x-b)(x+a\sqrt{-1})(x-a\sqrt{-1})$ -

$(x-a)(x-b+b\sqrt{-1})(x-b-b\sqrt{-1})$ $\left.\begin{matrix} \\ \\ \end{matrix}\right\} 2(m^2-n)(n^2-mp)-(mn-p)^2=0 \begin{cases} 4m^2-9n> \\ 4m^2-9n< \end{cases}$

$(x-b)(x-a+a\sqrt{-1})(x-a-a\sqrt{-1})$

$(x-b)(x-a+b\sqrt{-1})(x-a-b\sqrt{-1})$
$(x-a)(x-b+a\sqrt{-1})(x-b-a\sqrt{-1})$
$(x-a)(x+b+a\sqrt{-1})(x+b-a\sqrt{-1})$ $\left.\begin{matrix} \\ \\ \end{matrix}\right\} 100n(m^2+n)^2-(2m^3+17mn+p)^2=0 \begin{cases} 4m^2-9n> \\ 4m^2-9n< \\ m^2-n>0 \\ m^2-n<0 \end{cases}$

$(x-b)(x-b+a\sqrt{-1})(x-b-a\sqrt{-1})$
$(x-a)(x-a+b\sqrt{-1})(x-a-b\sqrt{-1})$ $\Big\}$ $2m^3-9mn+27p=0$ $\begin{cases} 4m^2-9n<0 \\ 4m^2-9n>0 \end{cases}$

$(x-a)(x-b)(x-c)$ - - - - - - $4(m^2-3n)(n^2-3mp)-(mn-9p)^2>0$

$(x-a)(x-b+c\sqrt{-1})(x-b-c\sqrt{-1})$ $\begin{cases} 4(m^2-3n)(n^2-3mp)-(mn-9p)^2<0 \\ 2(m^2-n)(n^2-mp)-(mn-p)^2>0 \\ 2m^3-9mn+27p>0, 4m^2-9n>0 \end{cases}$

$(x-a)(x+c+b\sqrt{-1})(x+c-b\sqrt{-1})$ $\begin{cases} mn-p>0, 2(m^2-n)(n^2-mp)-(mn-p)^2<0 \\ 100n(m^2+n)^2-(2m^3+17mn+p)^2<0 \end{cases}$

$(x-a)(x-c+b\sqrt{-1})(x-c-b\sqrt{-1})$ $mn-p<0, 100n(m^2+n)^2-(2m^3+17mn+p)^2<0$

$(x-b)(x-a+c\sqrt{-1})(x-a-c\sqrt{-1})$ $\begin{cases} 4(m^2-3n)(n^2-3mp)-(mn-9p)^2<0 \\ 100n(m^2+n)^2-(2m^3+17mn+p)^2<0 \\ 2m^3-9mn+27p<0, 4m^2-9n>0 \end{cases}$

$(x-b)(x+c+a\sqrt{-1})(x+c-a\sqrt{-1})$ $\begin{cases} mn-p<0, 100n(m^2+n)^2-(2m^3+17mn+p)^2>0 \\ (m^2-n)<0 \end{cases}$

$(x-b)(x-c+a\sqrt{-1})(x-c-a\sqrt{-1})$ $\begin{cases} mn-p>0, 2m^3-9mn+27p>0 \\ 100n(m^2+n)^2-(2m^3+17mn+p)^2>0, 4m^2-9n<0 \end{cases}$

$(x-c)(x-a+b\sqrt{-1}(x-a-b\sqrt{-1})$ $\begin{cases} 100n(m^2+n)^2-(2m^3+17mn+p)^2>0 \\ 2(m^2-n)(n^2-mp)-(mn-p)^2>0 \end{cases}$

$(x-c)(x-b+a\sqrt{-1})(x-b-a\sqrt{-1})$ $\begin{cases} 2m^3-9mn+27p<0 \\ 2(m^2-n)(n^2-mp)-(mn-p)^2<0, 4m^2-9n<0 \end{cases}$

## VII. $\left\{ x^3 - mx^2 - nx + p = 0 \right\}$

$(x+a)(x-a)(x-a)$ - - - - - - $m^2-n=0, m^3-p=0$

$(x+a)(x-b)(x-b)$
$(x+b)(x-a)(x-a)$ - - - - - $\Big\}$ $4(m^2+3n)(n^2+3mp)-(-mn+9p)^2=0$ $\begin{cases} m^2-n<0 \\ m^2-n>0 \end{cases}$

$(x-a)(x-b)(x+b)$
$(x+a)(x-a)(x-b)$ - - - - - $\Big\}$ $mn-p=0$ $\begin{cases} m^2-n>0 \\ m^2-n<0 \end{cases}$

$(x+a)(x-b+b\sqrt{-1})(x-b-b\sqrt{-1})$ $2(m^2+n)(n^2+mp)-(-mn+p)^2=0$

$(x+a)(x-a+b\sqrt{-1})(x-a-b\sqrt{-1})$ $2m^3-mn-p=0$

$(x+a)(x-b)(x-c)$ - - - - - $\begin{cases} 4(m^2+3n)(n^2+3mp)-(-mn+9p)^2>0 \\ mn-p<0, m^2-n<0 \end{cases}$

$(x-a)(x+b)(x-c)$ - - - - - - $mn-p>0$

$(x-a)(x-b)(x+c)$ - - - - - $\begin{cases} 4(m^2+3n)(n^2+3mp)-(-mn+9p)^2>0 \\ mn-p<0, m^2-n>0 \end{cases}$

$(x+a)(x-b+c\sqrt{-1})(x-b-c\sqrt{-1})$ $\begin{cases} 4(m^2-3n)(n^2+3mp)-(-mn+9p)^2<0 \\ 2(m^2+n)(n^2+mp)-(-mn+p)^2>0 \\ 2m^3-mn-p<0, m^2-n<0 \end{cases}$

$(x+a)(x-c+b\sqrt{-1})(x-c-b\sqrt{-1})$ $2(m^2+n)(n^2+mp)-(-mn+p)^2<0$

$(x+b)(x-a+c\sqrt{-1})(x-a-c\sqrt{-1})$ $\begin{cases} 4(m^2+3n)(n^2+3mp)-(-mn+9p)^2<0 \\ 2m^3-mn-p>0, m^2-n>0 \end{cases}$

VIII. $x^3+mx^2-nx+p=0$.

$(x+a)(x-b)(x-b)$ - - - - - - - $4(m^2+3n)(n^2-3mp)-(mn+9p)^2=0$
$(x+a)(x-b+b\sqrt{-1})(x-b-b\sqrt{-1})$ - - $2(m^2+n)(n^2-mp)-(mn+p)^2=0$
$(x+a)(x-b)(x-c)$ - - - - - - - $4(m^2+3n)(n^2-3mp)-(mn+9p)^2>0$
$(x+a)(x-b+c\sqrt{-1})(x-b-c\sqrt{-1}$ - $\begin{cases}4(m^2+3n)(n^2-3mp)-(mn+9p)^2<0\\2(m^2+n)(n^2-mp)-(mn+p)^2>0\end{cases}$
$(x+a)(x-c+b\sqrt{-1})(x-c-b\sqrt{-1})$ - - $2(m^2+n)(n^2-mp)-(mn+p)^2<0$

IX. $x^3-mx^2-nx-p=0$.

$(x-a)(x+b)(x+b)$ - - - - - - - - $4(m^2+3n)(n^2-3mp)-(mn+9p)^2$
$(x-a)(x+b+b\sqrt{-1})(x+b-b\sqrt{-1})$ - - $2(m^2+n)(n^2-mp)-(mn+p)^2=0$
$(x-a)(x+b)(x+c)$ - - - - - - - - $4(m^2+3n)(n^2-3mp)-(mn+9p)^2>0$
$(x-a)(x+b)+c\sqrt{-1})(x+b-c\sqrt{-1})$ - $\begin{cases}4(m^2+3n)(n^2-3mp)-(mn+9p)^2<0\\2(m^2+n)(n^2-mp)-(mn+p)^2>0\end{cases}$
$(x-a)(x+c+b\sqrt{-1})(x+c-b\sqrt{-1})$ - - $2(m^2+n)(n^2-mp)-(mn+p)^2<0$

$x^3+mx^2+p=(x+a)(x-c+b\sqrt{-1})(x-c-b\sqrt{-1})=0$

$x^3+mx^2-p=\begin{cases}(x-a)(x+a+a\sqrt{-1})(x+a-a\sqrt{-1}) & - - \; 2m^3-p=0\\(x+a)(x+a)(x-b) & - - - - - - - - \; 4m^3-27p=0\\(x+a)(x+b)(x-c) & - - - - - - - \; 4m^3-27p>0\\(x-a)(x+c+b\sqrt{-1})(x+c-b\sqrt{-1}) & - - \; 2m^3-p<0\\(x-b)(x+a+c\sqrt{-1})(x+a-c\sqrt{-1}) & - \; 4m^3-27p<0, 2m^3-p>0\end{cases}$

$x^3-mx^2+p=\begin{cases}(x+a)(x-a+a\sqrt{-1})(x-a-a\sqrt{-1}) & - \; 2m^3-p=0\\(x-a)(x-a)(x+b) & - - - - - - \; 4m^3-27p=0\\(x-a)(x-b)(x+c) & - - - - - \; 4m^3-27p>0\\(x+a)(x-c+b\sqrt{-1})(x-c-b\sqrt{-1}) & - \; 2m^3-p<0\\(x+b)(x-a+c\sqrt{-1})(x-a-c\sqrt{-1}) & - - \; 4m^3-27p<0, 2m^3-p>0\end{cases}$

$x^3-mx^2-p=(x-a)(x+c+b\sqrt{-1})(x+c-b\sqrt{-1})$

$x^3+nx+p=\begin{cases}(x+a)(x-b+a\sqrt{-1})(x-b-a\sqrt{-1}) & - - \; 100n^3-p^2=0\\(x+a)(x-c+b\sqrt{-1})(x-c-b\sqrt{-1}) & - - \; 100n^3-p^2<0\\(x+b)(x-c+a\sqrt{-1})(x-c-a\sqrt{-1}) & - \; 100n^3-p^2>0\end{cases}$

$x^3-nx+p=\begin{cases}(x+a)(x-b)(x-b) & - - - - - - - - \; 4n^3-27p^2=0\\(x+a)(x-b+b\sqrt{-1})(x-b-b\sqrt{-1}) & - - \; 2n^3-p^2=0\\(x+a)(x-b(x-c) & - - - - - - - - \; 4n^3-27p^2<0\\(x+a)(x-b+c\sqrt{-1})(x-b-c\sqrt{-1}) & - \begin{cases}4n^3-27p^2<0\\2n^3-p^2>0\end{cases}\\(x+a)(x-c+b\sqrt{-1})(x-c-b\sqrt{-1} & - - - \; 2n^3-p^2<0\end{cases}$

$x^3+nx-p=\begin{cases}(x-a)(x+b+a\sqrt{-1})(x+b-a\sqrt{-1}) & - - \; 100n^3-p^2=0\\(x-a)(x+c+b\sqrt{-1})(x+c-b\sqrt{-1}) & - - \; 100n^3-p^2<0\\(x-b)(x+c+a\sqrt{-1})(x+c-a\sqrt{-1}) & - - \; 100n^3-p^2>0\end{cases}$

$x^3-nx-p=\begin{cases}(x-a)(x+b)(x+b) & - - - - - - \; 4n^3-27p^2=0\\(x-a)(x+b+b\sqrt{-1})(x+b-b\sqrt{-1}) & - - \; 2n^3-p^2=0\\(x-a)(x+b)(x+c) & - - - - - - - \; 4n^3-27p^2>0\\(x-a)(x+b+c\sqrt{-1})(x+b-c\sqrt{-1}) & - \begin{cases}4n^3-27p^2<0\\2n^3-p^2>0\end{cases}\\(x-a)(x+c+b\sqrt{-1})(x+c-b\sqrt{-1}) & - \; 2n^3-p^2<0\end{cases}$

$x^3+p=(x+a)(x-c+b\sqrt{-1})(x-c-b\sqrt{-1})$; $x^3-p=(x-a)(x+c+b\sqrt{-1})(x-c-b\sqrt{-1})$

XX.

If one of the Roots of $x^m + px^{m-1} + qx^{m-2} \ldots\ldots + r = 0$, is an integer Number $a$, either positive or negative, it will be a Divisor of the last Term $r$; for $a^m + pa^{m-1} + qa^{m-2} \ldots + na + r = 0$, consequently $a^m + pa^{m-1} + qa^{m-2} \ldots + na = -r$, wherefore $a^{m-1} + pa^{m-2} + qa^{m-3} \ldots + n = -\frac{r}{a}$. The first Member of which Equation is an Integer, as being composed of Integers, wherefore $\frac{r}{a}$ is an Integer, wherefore $a$ is one of the Divisors of $r$. Whence is derived a Method for investigating all the commensurable Roots of an Equation; for they should all be found by trying the Division of the Equation, by $x$ more or less each of the Divisors of the last Term. Method of finding the commensurable roots of an equation.

For Example, suppose the Equation $x^3 - 5x^2 + 7x - 3 = 0$ is to be resolved, the Divisors of the last Term $-3$, being either $-1$, or $-3$, or $+1$, or $+3$; I try the Division by $x - 1$, $x - 3$, $x + 1$, $x + 3$; the Division by $x - 1$ and $x - 3$ succeeding, I perceive that the Equation may be wrote thus, $(x - 1) \times (x - 1) \times (x - 3) = 0$, whereby it appears that the proposed Equation has three commensurable Roots, one of which is $+3$, and the other two are equal, and expressed each by $+1$.

When an Equation is not divisible by a simple Equation composed of $x +$ or $-$ some one of the Divisors of the last Term, it will be a Sign that this Equation has no commensurable Root.

XXI.

It may be objected against the foregoing Method of finding the commensurable Roots of an Equation, that if any of the Roots of this Equation is a Fraction, it will not be possible to find it among the Divisors of the last Term: because if fractional Divisors be admitted in a Number, such Divisors may be found without End. But this Objection is easily removed, by observing that in an Equation clear of Fractions, and having the Coeficient of its highest Term Unit, the Value of the unknown Quantity cannot be a Fraction $\left(\frac{a}{b}\right)$, whose Numerator and Denominator are rational and integer Numbers. In an equation whose coeficients are Integers the unknown quantity cannot be a fraction.

Let for Example $x^3 + px^2 + qx + r = 0$, and let $\frac{a}{b}$ be supposed to be a Root of this Equation, then $\frac{a^3}{b^3} + \frac{pa^2}{b^2} + \frac{qa}{b} + r = 0$, and $a^3 + pba^2 + qb^2a^2 + b^3r = 0$, consequently by the foregoing Article the Integer Number $a$ should be a Divisor of the last Term $b^3r$. Now,

since, $a$ and $b$ have no common Diviſor, the Fraction $\frac{a}{b}$ being by Hypotheſis, reduced to its loweſt Terms, it follows that $a$ and $b^3$ have no common Diviſor, Art. XXIX. Chap. II. Wherefore $a$ muſt be a Diviſor of $r$, wherefore $r = n\,a$, $n$ being an integer Number; whence we have $a^3 + p\,b\,a^2 + q\,b^2\,a + n\,b^3\,a = 0$, wherefore $a^2 + p\,b\,a + q\,b^2 + n\,b^3 = 0$, therefore, by the foregoing Article, $a$ ſhould be a Diviſor of the laſt Term $q\,b^2 + n\,b^3$, and conſequently of $q + b\,n$, wherefore $q + b\,n = m\,a$, whence $a^2 + p\,b\,a + b^2\,m\,a = 0$, and $a + p\,b + b^2\,m = 0$, and of courſe $\frac{a}{b} = -p - m\,b$, that is, $=$ an integer Number which is contrary to the Hypotheſis.

XXII.

Transformation by means of which an equation may be cleared of fractions.

When a Problem produces an Equation whoſe Coeficients are Fractions, the commenſurable Roots it may contain, cannot be found by the foregoing Method; however the Problem by a ſimple Transformation may be changed into another, in which the Equation to be reſolved is clear of Fractions, and having the Coeficient of its higheſt Term, Unit.

Suppoſe the Equation propoſed is $x^3 + \frac{a}{b}x^2 + \frac{c}{d}x + \frac{e}{f} = 0$, by putting the unknown Quantity $x$ equal to another $y$ divided by ſome undetermined Number $m$, I transform the Equation into a new one, $\frac{y^3}{m^3} + \frac{a\,y^2}{b\,m^2} + \frac{c\,y}{d\,m} + \frac{e}{f} = 0$, or $y^3 + \frac{a\,m}{b}y^2 + \frac{c\,m^2}{d}y + \frac{e\,m^3}{f} = 0$, in which I perceive, that if $m$ is diviſible by $b$, by $d$, and by $f$; $\frac{a\,m}{b}$, $\frac{c\,m^2}{d}$ and $\frac{e\,m^3}{f}$ will be Integers, we have therefore no more to do than to aſſume for $m$ the Product of the Numbers $b$, $d$, $f$, and if thoſe Numbers are not prime to each other, it will be eaſy to find a Number leſs than their Product that will be diviſible by all the three.

The Equation being transformed into another clear of Fractions, the commenſurable Roots of this laſt Equation are to be inveſtigated by the foregoing Method; and if none are found, neither will the firſt have any, ſince $x$ ſuppoſed commenſurable, cannot give an incommenſurable Quantity when divided by the Number $m$, which is alſo commenſurable.

XXIII.

Inconvenicency to which the foregoing method is liable.

The foregoing Method is liable to this Inconveniency, that when the laſt Term has very many Diviſors, the Labour will be exceſſive to try all the Diviſions that this Method preſcribes; how the Analyſts have abridged it, by limiting to a ſmall Number the Diviſors that are to be tried, we ſhall now proceed to explain.

XXIV.

They obſerved firſt, that if in the Root $x + a$ of any Equation, or if in the Diviſor $x + a$ of any Quantity, $x$ be put equal to a given Number, the Number into which the Diviſor will then be transformed, will be a Diviſor of the propoſed Quantity, in which $x$ has been put equal to the ſame Number, that is, for Example, if in the Quantity $x^3 - 2x - 21$ of which $x-3$ is a Diviſor, $x$ be put $= 5$, the Number 94 into which $x^3 - 2x - 21$ is transformed by this Suppoſition is neceſſarily diviſible by the Number 2 into which $x-3$ is transformed by the ſame Suppoſition. Obſervations that have ſerved to render the foregoing method more perfect.

They then ſuppoſed $x$ in the Quantity whoſe Diviſor they ſought, ſucceſſively equal to ſeveral Numbers, ſuch, for Example, as $+1, 0, -1$; Suppoſitions that ſhould be firſt made, becauſe they are the eaſieſt ſubſtituted, they afterwards ſought the Diviſors of the Numbers into which the propoſed Quantity is transformed by thoſe Subſtitutions, and obſerved:

1°. That among all the Diviſors of the Number ariſing from the Suppoſition of $x = +1$ in the propoſed Quantity, the Number $1 + a$ ſhould be found, ſince $x + a$ is the Diviſor ſought.

2°. That among all the Diviſors of the Number ariſing from the Suppoſition of $x = 0$, which are no other than the Diviſors of the laſt Term of the propoſed Quantity, the Number $a$ ſhould be found.

3°. That among all the Diviſors of the Number ariſing from the Suppoſition of $x = -1$ ſhould be the Number $-1 + a$.

XXV.

Now as the Numbers $1 + a, a, -1 + a$ are neceſſarily ſuch, that the firſt exceeds the ſecond by Unity, and that the ſecond exceeds the third alſo by Unity; it follows that of all the Diviſors of the Number ariſing from the Suppoſition of $x = 0$, thoſe only can expreſs $a$, that are exceeded by Unity by ſome one of the Diviſors of the Number ariſing from the Suppoſition of $x = 1$, and exceed at the ſame Time by Unity ſome one of the Diviſors of the Number ariſing from the Suppoſition of $x = -1$. Fundamental principle for finding the commenſurable roots of an equation.

If ſeveral Numbers are found among the Diviſors ariſing from the Suppoſition of $x = 0$, that have the above mentioned Conditions, to reduce the Diviſors of the laſt Term to ſtill more narrow Limits, $x$ is to be put $= 2$, and then obſerve if among the Diviſors of the Number ariſing from this Suppoſition, there be found any that exceed by Unity thoſe ariſing from the Suppoſition of $x = 1$, and ſo on.

It is eaſy to perceive that the Trial that is to be made of all thoſe Diviſors ſhould be double, that is, that each of them ſhould be taken as well negatively as poſitively.

To illuſtrate this Method and to render the Application of it more eaſy, we ſhall proceed to give ſome Examples ſhewing the Order to be obſerved in the Calculation to avoid miſtakes, and to abridge, as much as poſſible, the Labour of the Calculator.

XXVI.

Let it be propoſed to find the commenſurable Roots of the Equation $x^3 - 2x^2 - 13x + 6 = 0$; or which is the ſame, let it be propoſed to find the Diviſors of one Dimenſion of the Quantity

$$x^3 - 2x^2 - 13x + 6$$

| | | | | |
|---|---|---|---|---|
| 1 | 8 | 1 . 2 . 4 . 8 | +4 | −2 |
| 0 | 6 | 1 . 2 . 3 . 6 | +3 | −3 |
| −1 | 16 | 1 . 2 . 4 . 8 . 16 | +2 | −4 |
| −2 | 16 | | +1 | |

Application of the foregoing method to an example.

I firſt ſet down one under the other the Suppoſitions 1, 0, − 1, to be made for $x$; I ſet down afterwards beſide each of thoſe Numbers, the Numbers − 8, + 6, + 16, or ſimply 8, 6, 16, (becauſe the Signs do not affect the Diviſors) into which the Quantity $x^3 - 2x^2 - 13x + 6$ is transformed by thoſe Suppoſitions, and I ſeparate them from the firſt Numbers by a vertical Line. I ſet down in a third Column the Numbers 1, 2, 4, 8; 1, 2, 3, 6; 1, 2, 4, 8, 16, which are the Diviſors of the foregoing Numbers; the four firſt beſide 8, of which they are the Diviſors; the four that follow beſide 6, and the five laſt beſide 16.

This being done, to find among the Diviſors, 1, 2, 3, 6, of the Number 6 ariſing from the Suppoſition of $x = 0$, that which is to be added or deducted from $x$ to obtain the Diviſor ſought, or rather to exclude from all thoſe Diviſors, thoſe that have not the required Conditions; I obſerve 1°. that 1, which is the firſt of thoſe Diviſors, cannot be admitted, whether it be taken with the Sign + or with the Sign −; for if it be taken with the Sign +, that is, if $x + 1$ be conſidered as the Diviſor ſought, this Diviſor becoming 2 by the Suppoſition of $x = +1$, and 0 by the Suppoſition of $x = -1$, we ſhould find 2 among the Numbers of the firſt Line, and 0 among thoſe of the third: Now the ſecond of thoſe Conditions is not anſwered. As to − 1, it cannot be admitted either; that is, $x - 1$ is not the Diviſor ſought, for this Diviſor becoming 0 by the Suppoſition of $x = +1$, and − 2 by the Suppoſition of $x = -1$, we ſhould find 0 among the Numbers of the firſt Line, and the Number 2 among thoſe of the ſecond. Now only the ſecond of thoſe two Conditions is anſwered. I next perceive that the Diviſor 2 is alſo to be rejected, becauſe if it be taken with the Sign +, that is, if $x + 2$ be conſidered as the Diviſor ſought, we would have + 3 by the Suppoſition of $x = +1$, and + 1, by the Suppoſition of $x = -1$, which would require that the Number 3 ſhould be found in the firſt Line, and 1 in the third: Now the firſt of thoſe two Conditions is not fulfilled. And if 2 be taken with the Sign −, that is, if $x - 2$ be conſidered as the Diviſor, we would have − 1 and − 3 for the Suppoſitions of $x = +1$ and of $x = -1$, which

would require that 1 ſhould be found in the firſt Line and in the third: the firſt of which Conditions is only fulfilled.

Having excluded 1 and 2, I aſſume the Diviſor 3, and I perceive that taking it with the Sign +, that is, conſidering $x + 3$ as the Diviſor ſought, we ſhould find $+4$ by the Suppoſition of $x = +1$, and $+2$ by the Suppoſition of $x = -1$: and in effect I find 4 in the firſt Line and 2 in the third, wherefore $+3$ has the Conditions required, I therefore ſet it down in the ſecond Line, that is, oppoſite the Number of which it is a Diviſor. I ſet down at the ſame Time the Numbers $+4$ and $+2$ in the Line above it, and below it, not that thoſe Numbers are to be joined to $x$ to ſerve as Diviſors to the propoſed Quantity, but becauſe the Trial of the Diviſors not being compleated, other Numbers beſides $+3$ might be found having the required Conditions, and in this Caſe new Suppoſitions are to be made, to diſcover among thoſe Numbers thoſe that are to be excluded. I now try if 3 taken with the Sign — will ſucceed, that is, if $x - 3$ has the Conditions required to be a Diviſor of the propoſed Quantity; in this Caſe we ſhould have $-2$ and $-4$ by the Suppſitions of $x = +1$, and of $x = -1$, which happens in effect, whence $x - 3$ has the Conditions required to be a Diviſor as well as $x + 3$, I therefore ſet down in a fifth vertical Column $-2$, $-3$, $-4$.

Laſtly, I try 6, and I perceive that if it be taken with the Sign + we ſhould find $+7$ and $+5$ in the Lines above it and below it, which does not happen; and if it be taken with the Sign — we ſhould find $-5$ and $-7$ in the ſame Lines, which likewiſe does not happen. I conclude therefore that the propoſed Quantity can have but $x - 3$ and $x + 3$, for Diviſors of one Dimenſion.

To diſcover if there is the ſame Foundation for trying the Diviſion by $x - 3$ as by $x + 3$; I obſerve that if a fourth Line was made by ſuppoſing $x = -2$, we ſhould find $-5$ for the fourth Term of the Column $-2$, $-3$, $-4$; and $+1$ for the fourth Term of the Column $+4$, $+3$, $+2$; for it is manifeſt that the Diviſor $x - 3$, would become $-5$, by the Suppoſition of $x = -2$, and that the Diviſor $x + 3$ would become $+1$ by the ſame Supppoſition. But by putting $x = -2$ in the propoſed Quantity $x^3 - 2x^2 - 13x + 6$, it becomes 16 which is not diviſible by 5, and is by 1. Wherefore $x - 3$ cannot be a Diviſor of this Quantity, whence if it has one, it muſt be $x + 3$, I therefore divide $x^3 - 2x^2 - 13x + 6$ by $x + 3$ which ſucceeds, and gives for exact Quotient $xx - 5x + 2$.

In Numbers ſo ſimple as 8, 6, 16, it was an eaſy Matter not to omit any of their Diviſors, becauſe thoſe Numbers have but few Diviſors, but in Numbers that have many Diviſors, to obtain them all, they muſt be ſought with a certain Order, we ſhall therefore proceed to explain by an Example how the Analyſts perform this Operation.

## XXVII.

Method of finding all the diviſors of any propoſed number.

Let it be propoſed to find all the Diviſors of the number 120.

| | | | | | | | | I firſt | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | 1 | | 120 |
| | | | | | | | 2 | | 60 |
| | | | | | | 4. | 2 | | 30 |
| | | | | | 8. | 4. | 2 | | 15 |
| | | | | 24. | 12. | 6. | 3 | | 5 |
| 120. | 60. | 40. | 30. | 20. | 15. | 10. | 5 | | 1 |

draw a vertical Line to the left Hand of this Number, and to the left Hand of this Line oppoſite to 120 I ſet down 1, as being its firſt Diviſor. I afterwards try whether 2 will divide 120, and as the Diviſion ſucceeds, I ſet down 2 to the left Hand of the vertical Line, oppoſite to 60 the Quotient of the Diviſion, which I ſet down to the right Hand of the ſame Line.

I afterwards try the Diviſion by 2, which ſucceeds, and gives 30 for Quotient, I then ſet down the new Diviſor 2 under the firſt, and 30 under 60. I multiply at the ſame Time the new Diviſor 2 by the one above it 2, and ſet down the Product 4 to the left Hand of the ſecond 2, as being a new Diviſor of the propoſed Number 120.

As 30 is diviſible by 2, I ſet down 2 again to the left Hand of the vertical Line in the fourth Row, and the Quotient 15 to the right Hand oppoſite to it in the ſame Row. I multiply at the ſame Time the new Diviſor 2 by 4, which gives 8 for a new Diviſor of the propoſed Number. But omit multiplying it by the firſt Diviſors, becauſe there would reſult 4, which has been already ſet down.

15 not being diviſible by 2, I try the Diviſion by 3, which ſucceeds, and gives 5 for the Quotient, which I ſet down to the right Hand in the fifth Row, and the Diviſor 3 to the left; I afterwards multiply 3 by 2, by 4 and by 8, which I find in the ſuperior Lines, and ſet down to the left Hand of 3, the Products 6, 12, 24, which are, as is manifeſt, new Diviſors of the propoſed Number.

5 having no Diviſor I ſet it down to the left of the vertical Line in the fifth Row, and I ſet down at the ſame Time the Product of this Number into all the foregoing Diviſors 2, 3, 4, 6, 8, 12, 24, and there reſults 10, 15, 20, 30, 40, 60, 120, which I ſet down in the ſame Row to the left Hand of 5.

Which being done, all the Numbers which are to the left Hand of the vertical Line from 1 to 120 are the Diviſors of 120. After the ſame Manner all the Diviſors of any other Number may be found by dividing it by its leaſt Diviſor that exceeds Unit, and the Quotient again by its leaſt Diviſor, and ſo on, till you have a Quotient that is not diviſible by any Number greater than Unit. This Quotient, with theſe Diviſors, will be the firſt or ſimple Diviſors of the Number, and the Products of the Multiplication of any 2, 3, 4, &c. of them will be the compound Diviſors.

XXVIII.

Let it now be propoſed to find the commenſurable Roots of the Equation.

$$x^5 - 12x^4 + 5x^3 - 61x^2 + 22x - 120 = 0.$$

| | | | | |
|---|---|---|---|---|
| 1 | 165 | 1 . 3 . 5 . 11 . 15 . 33 . 55 . 165 | | |
| 0 | 120 | 1 . 2 . 3 . 4 . 5 . 6 . 8 . 10 . 12 . 15 . 20 . 24 . 30 . 40 . 60 . 120 | +3 | —11 |
| — 1 | 221 | 1 . 13 . 27 . 221 | +2 | —12 |
| | | | +1 | —13 |

Another example of finding the commenſurable roots of an equation by the foregoing method.

Having ſet down in a firſt vertical Column 1, 0, — 1 for the Values to be ſucceſſively given to $x$; and in another vertical Column the Numbers 165, 120, 221 reſulting from the Subſtitution of thoſe Values in the Quantity $x^5 - 12x^4 + 5x^3 - 61x^2 + 22x - 120$. I place in a third Column the Diviſors of thoſe three Numbers, each oppoſite the Number that produced them. This being done, I examine firſt if among the Diviſors of 120 the Number 2 has the required Conditions, and I perceive that when taken with the Sign + it agrees with the Numbers 3 and 1 above and below it. I therefore ſet down in the fourth vertical Column + 3, + 2, + 1. But the ſame Number taken with theSign — does not ſucceed, as — 3 is not found below it.

Examining in like Manner all the other Diviſors of 120, I obſerve, that the Number 12 taken with the Sign — has the required Conditions, provided the Numbers 11 and 13 that are above and below it are alſo taken with the Sign —, I therefore ſet down in a fifth vertical Column the three Numbers — 11, — 12, — 13.

To diſcover now by which of the two Quantities $x + 2$ or $x - 12$ the Diviſion is to be tried, I obſerve, that if it was by the firſt, — 2 ſubſtituted for $x$ in the Equation ſhould make it vaniſh, and as this does not happen, I conclude that the Diviſion by $x - 12$ is the only one to be tried, which ſucceeds and gives $x^4 + 5x^2 - x + 10$ for Quotient, whence 12 is one of the Values of $x$ in the given Equation, and the only commenſurable one.

XXIX.

Let it be propoſed to find the commenſurable Roots of the Equation.

$$x^6 - 4x^5 + 3x^4 - 12x^3 - 5x^2 + 11x + 36 = 0$$

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | — 1 | — 1 | |
| 2 | 74 | | | — 1 | — 2 | |
| 1 | 30 | 1 . 2 . 3 . 5 . 10 . 15 . 30 | + 3 | — 2 | — 3 | + 10 |
| 0 | 36 | 1 . 2 . 3 . 4 . 6 . 9 . 12 . 18 . 36 | + 2 | — 3 | — 4 | + 9 |
| — 1 | 40 | 1 . 2 . 4 . 5 . 8 . 10 . 20 . 40 | + 1 | — 4 | — 5 | + 8 |

Third application of the method of finding the commenſurable roots in an equation.

Having ſet down in a firſt vertical Column the Values 1, 0, — 1 to be given to $x$; in a ſecond, the Numbers 30, 36, 40, into which the propoſed Quantity is transformed by theſe Suppoſitions, and in a third, all the Diviſors 1, 2, 3, 5, 6, 10, 15, 30 of the Number 30; the Di-

viſors 1, 2, 3, 4, 6, 9, 12, 18, 36 of the Number 36, the Diviſors 1, 2, 4, 5, 8, 10, 20, 40 of the Number 40; I find among all thoſe Diviſors four Columns that have the required Conditions, the firſt + 3, + 2, + 1, the ſecond — 2, — 3, — 5, the third — 3, —4, — 5, the fourth + 10, + 9, + 8.

To diſcover which of thoſe Progreſſions are to be rejected, I firſt put $x = 2$, and ſet down 2 above 1 in the firſt Column, I afterwards ſet down in the ſecond Column 74, the Number into which the propoſed Quantity is transformed by ſuppoſing $x = 2$. I now perceive without taking the Trouble of inveſtigating the Diviſors of this Number, that the two Columns + 3, + 2, + 1, and + 10, + 9, + 8, are to be rejected, for if the firſt could be admitted, we ſhould find + 4 among the Diviſors of 74, and if the ſecond could be admitted, we ſhould find + 11 among the ſame Diviſors.

But I perceive, that the Numbers — 1 and — 2, that the Columns — 2, — 3, — 4, and — 3, — 4, — 5 require, are Diviſors of 74. I therefore ſet down the Numbers — 1 and — 2 above — 2 and — 3 in the ſecond and third Colums, I afterwards try to exclude one of theſe two Columns — 1, — 2, — 3, — 4; and — 2, — 3, — 4, — 5; which is eaſily effected, ſince if the firſt was to be admitted, the Equation ſhould be reduced to 0 by putting $x = 3$, which not happening, there remains only the Column — 1, — 2, — 3, — 4, — 5 to be tried, that is, the only Diviſor to be tried is $x - 4$, which ſucceeds, and gives for Quotient $x^5 + 3x^3 - 5x - 9$. whence + 4 is one of the Values of $x$ in the propoſed Equation, and the only commenſurable one.

XXX.

Advantage of inveſtigating the commenſurable diviſors in the reduced rather than in the propoſed equation of the fourth degree.

When an Equation of the fourth Degree has been found to have its four Roots real, before its reduced Equation is attempted to be ſolved by Approximation, in order to obtain the Value of $x$ to ſubſtitute it in that of $z$ (Art. CL. Chap. III.); it will be proper to examine by the foregoing Method whether any of its Roots are commenſurable if it has but one commenſurable Root, the other thre cannot be found otherwiſe than by Approximation, *viz.* by ſolving an Equation of the third Degree, whoſe three Roots are real; but if an Equation of the fourth Degree has two commenſurable Roots, its Reſolution may be compleatly effected, ſince when theſe two Roots are found, there will only remain to be ſolved an Equation of the ſecond Degree.

XXXI.

When an Equation of the fourth Degree has two commenſurable Roots it will be eaſier to find them by Means of the reduced Equation than by itſelf; for it is manifeſt that in this Caſe, in the four Values of $z$ repreſented in general by $i + k$; $i - k$; $-i + l$; $-i - l$; $i$ can only be a commenſurable Quantity, and conſequently in the Root $xx - 4ii$

of the reduced Equation, 4 *ii* will repreſent a commenſurable and Square Quantity, whence we are to ſelect out of the Diviſors of the laſt Term, only ſquare Quantities, affected with the Sign —.

Likewiſe if the Equation $z^4 + pzz + qz + r = 0$ can be reſolved into two Equations of the ſecond Degree the Coeficients of whoſe Terms are commenſurable Quantities, we are to inveſtigate the Diviſors of the reduced Equation, and ſelect out of the Diviſors of the laſt Term only ſquare Quantities, and affected with the Sign —.

XXXII.

How to diſtinguiſh the equations of the fourth degree whoſe roots are only affected by ſquare radicals.

Tho' an Equation of the fourth Degree has not two commenſurable Roots, or is not reſolvable into two Equations of the ſecond Degree, the Coeficients of whoſe Terms are commenſurable Quantities, it may be reſolvable into two Equations of the ſecond Degree, the Coeficients of whoſe Terms are ſquare Radicals, in this Caſe the Part $\pm \frac{1}{2} x$ of the Expreſſion $\pm \frac{1}{2} x + \sqrt{\left(-\frac{1}{4}xx - \frac{1}{2}p \mp \frac{q}{2x}\right)}$ of the Value of $z$, or which is the ſame, the Part $i$ common to the four Roots $i + k$, $i - k$, $-i + l$, $-i - l$, ſhould be a ſimple Radical of the ſecond Degree, becauſe in the reduced Equation $x$ is every where of an even Number of Dimenſions, conſequently in all Equations of this Nature, their reduced Equation is diviſible by $xx \pm$ a commenſurable Quantity, and of courſe Equations of the fourth Degree are reſolvable by the foregoing Method, when their reduced Equation has Diviſors of this Form.

XXXIII.

The foregoing obſervations illuſtrated by an example.

Let, for Example, the Equation of Art. CLI. Chap. III. $u^4 + 16u^3 + 99u^2 + 228u + 144 = 0$ be propoſed to be ſolved, to diſcover whether it has two rational Roots, I ſelect out of the Diviſors of the laſt Term 2704 of its reduced Equation, $x^6 + 6x^4 - 183x^2 - 2704 = 0$, thoſe that are Squares, and that are affected with the Sign —, and I find that this reduced Equation has for Diviſor $xx - 16 = 0$, whence the Roots of the propoſed Equation are found as in the above mentioned Article.

In like Manner, if the Equation of Art. CLIV. Chap. III. $z^4 - 6z^2 + 8z - 1 = 0$ was propoſed to be ſolved, it would appear by this Method that its reduced Equation $x^6 - 12x^4 + 40x^2 - 64 = 0$, has for Diviſor $xx - 8$, whence are obtained the Roots of the propoſed Equation.

To ſhow the Uſe of the foregoing Rules in the Solution of Problems, here follow ſome Examples:

XXXIV.

*Two Maſons* A *and* B *jointly perform a Piece of Work in* 12 *Days; now, if the Sum of the Days in which they could each have ſeparately performed the ſame, be multiplied by the Days in which* A *alone (he working quicker than* B*) could have done it, the Product will be* 1000, *in what Time could each do it?*

Another problem ſolved by the foregoing method.

Suppoſe *A* could do it alone in $x$ Days, and *B* in $y$ Days, then $x : 1 = 12 : \frac{12}{x}$ = the Work done by *A*, and $y : 1 = 12 : \frac{12}{y}$ = the Work done by *B*, in 12 Days: then $\frac{12}{y} + \frac{12}{x} = 1$, or $12x + 12y = xy$, or $12x = xy - 12y$, then $y = \frac{12x}{x - 12}$, conſequently $\frac{12xx}{x - 12} = xy$, but $(x + y) \times x = xx + xy = 1000$, that is, $xx + \frac{12xx}{x - 12} = 1000$, or $x^3 - 12xx + 12xx = 1000x - 12000$, then $x^3 - 1000x + 12000 = 0$. Now, by the foregoing Method, there will be found the Diviſors 21, 20, 19, differing by Unity, but $x - 20$ will divide the Equation, wherefore $x = 20$, and $y = 30$.

XXXV.

*A Maſon had two cubical Pieces of Marble, containing between them* 1072 *ſolid Inches, and being ſeverally placed in his Yard, they ſtand upon* 130 *ſquare Inches of the Ground, what are the Sides of thoſe Cubes?*

Third problem ſolved by the foregoing method.

Suppoſe $x =$ the Side of the greater, and $y$ the Side of the leſſer Cube, then $x^3 + y^3 = 1072$; or $x^3 = 1072 - y^3$ and $xx + yy = 130$, and $x^2 = 130 - yy$; but $x^6 = (1072 - y^3)^2 = (130 - yy)^3$; hence $y^6 - 195y^4 - 1072y^3 + 25350y^2 - 523908 = 0$; and by the foregoing Method, there will be found among the Diviſors 9, 8, 7, 6, 5, differing by Unity, but $y - 7$ will divide the Equation, wherefore $y = 7$ and $x = 9$.

XXXVI.

If the propoſed Equation has no ſimple Diviſor, then we are to enquire if it has not ſome quadratick Diviſor, (if itſelf is an Equation of more than three Dimenſions) the Solution of Equations of the ſecond Degree being as compleat as that of the firſt. How the Analyſts have proceeded in this Reſearch we ſhall therefore continue to explain.

Inveſtigation of the method of finding the quadratick commenſurable factors of any propoſed equation.

Let $xx + bx + c = 0$ expreſs the Equation of the ſecond Degree that may be one of the Factors of a given Equation, or what amounts to the ſame, let $xx + bx + c$ be the Diviſor ſought of a given Quantity, by putting $x = 0$ in this Diviſor, it is manifeſt, that it will be reduced to the Number $c$, and that this Number will be one of the Diviſors of the laſt Term of the given Quantity.

If we afterwards put $x = +1$ in the Diviſor $x^2 + b + xc$, it will be transformed into $1 + b + c$, which will be one of the Diviſors of the Number that reſults, by putting likewiſe in the propoſed Quantity $x = 1$. Wherefore, if all the Diviſors of this Number be inveſtigated, and Unity be deducted from each of them affected as well with the Sign $+$ as with the Sign $-$, among the Numbers poſitive and negative, reſulting from this Operation, the Number expreſſing $b + c$ will be found.

If laſtly we put $x = -1$, as well in the propoſed Quantity as in the Diviſor $xx + bx + c$, which then becomes $1 - b + c$, it is eaſy to perceive that if all the Diviſors of the Number that reſults be inveſtigated, and Unity be deducted from each of them affected as well with the Sign $+$ as with the Sign $-$, among the Numbers poſitive and negative reſulting from this Operation, the Number expreſſing $-b + c$ will be found.

Now, as $c$ is an arithmetical Mean between $b + c$ and $-b + c$, it follows that out of the three Series of Numbers expreſſing $b + c$, $c$, $-b + c$ thoſe only that are in arithmetical Progreſſion are to be ſelected. When three Numbers in arithmetical Progreſſion are found, it is manifeſt the one correſponding to the Suppoſition of $x = 0$ is to be aſſumed for $c$, and as the Number that correſponds to the Suppoſition of $x = 1$, will repreſent $b + c$, it is manifeſt, that by deducting the firſt from the ſecond, the Quantity $b$ will be obtained; ſubſtituting afterwards thoſe two Numbers in the Room of $b$ and $c$ in $x^2 + bx + c$, there will reſult a Diviſor for the propoſed Quantity to be tried, and the only one to be tried, if but one arithmetical Progreſſion has been found among all the Series of Numbers furniſhed by the Diviſors of the Quantity in which $x$ has been made ſucceſſively $= 1, 0, -1$. If ſeveral arithmetical Progreſſions have been found, thoſe that are to be rejected will be found much in the ſame Manner as in the Caſe of ſimple Diviſors, by making new Suppoſitions for $x$, as $-2$, $-3$, or $+2$, $+3$, &c.

For if in the propoſed Quantity we put, for Example, $x = -2$, $x = -3$, &c. it is manifeſt that all the Diviſors, as well poſitive as negative of the Number that reſults, will be expreſſed by $4 - 2b + c$, or $9 - 3b + c$, &c. into which $x^2 + bx + c$ is transformed, when $x = -2$, or $x = -3$, &c. and deducting from all thoſe ſame Diviſors 4, 9, &c. the remainders will be expreſſed by $-2b + c$, $-3b + c$, &c. Now, $-2b + c$, $-3b + c$, &c. being Terms of the arithmetical Progreſſion $b + c$, $c$, $-b + c$, no more remains to be done than to ſearch among the Progreſſions already found, for one that is not altered by theſe new Values of $x$, which will ſerve to determine the Numbers $b$ and $c$.

XXXVII.

Let it be propoſed, for Example, to find a Diviſor of two Dimenſions of the Quantity

$$x^5 + 3x^4 + 2x^3 + 8x^2 - 36x + 21.$$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | — 2, 0 | — 2 | —2 | 0 | 0 | 0 |
| 0 | 21 | 1. 3. 7. 21 | 0 | —21, —7, —3, —1, 1, 3, 7, 21 | — 1 | 1 | — 7 | — 3 | —1 |
| —1 | 65 | 1. 5. 13. 65 | 1 | —66, —14, —6, —2, 0, 4, 12, 64 | 0 | 4 | —14 | — 6 | —2 |
| —2 | 125 | 1. 5. 25. 125 | 4 | —129, —29, —9, —5, —3, 1, 21, 121 | 1 | | | — 9 | —3 |
| —3 | 147 | 1. 3. 7. 21. 49. 147 | 9 | —156, —58, —30, —16, —12, —10, —8, —6, —2, 12, 40, 138 | | | | —12 | |

I firſt ſet down in a vertical Column the Values 1, 0, — 1 that are propoſed to be given to $x$, I ſet down afterwards in a ſecond vertical Column the Numbers 1, 21, 65, into which the propoſed Quantity is tranſformed by thoſe Suppoſitions.

Application of the foregoing method to an example.

I ſet down in like Manner in a third Column, beſide the firſt Number, its only Diviſor 1; beſide 21 its Diviſors 1, 3, 7, 21; and beſide 65 its Diviſors 1, 5, 13, 65.

This being done, I ſet down in a fourth Column the Numbers 1, 0, 1, Squares of 1, 0, — 1 wrote in the firſt Column, and conſequently the Values of $xx$ in the ſame Suppoſitions of 1, 0, — 1. Laſtly, I form a fifth Column from thoſe Conditions:

1°. That the firſt Line may be — 2, 0, found by deducting 1 from 1 taken with the Sign — and with the Sign +.

2°. That the ſecond Line may be compoſed of the Numbers — 21, — 7, — 3, — 1, + 1, + 7, + 21, the ſame as the Diviſors which are ſet down beſide them, but taken with the Sign — and with the Sign +.

3°. That the Numbers of the third Line may be — 66, — 14, — 6, — 2, 0, + 4, + 12, + 64, the firſt of which — 66, — 14, — 6, — 2 found by deducting 1 from the Numbers 65, 13, 5, 1 affected with the Sign —, and the others 0, + 4, + 12, + 64 found by deducting 1 from the ſame Numbers 1, 5, 13, 65 affected with the Sign +.

To determine afterwards the arithmetical Progreſſions contained in thoſe three ſeries of Numbers. I begin by taking in the firſt Line the Number — 2 for the firſt Term of a Progreſſion, I take ſucceſſively for ſecond Terms all thoſe of the ſecond Line; and ſeek at the ſame Time the third Terms which theſe two firſt would require, and I examine which of thoſe third Terms are found in the third Line: Now, — 2 and — 21 ſhould give for third Term — 40, which is not found in the third Line, I therefore reject 21, and next aſſume — 7 for ſecond Term, and as it ſhould give — 12 for third Term, and that — 12 is not found in the third Line, I reject alſo — 7, as likewiſe — 3, becauſe this laſt ſhould give — 4, which is not found in the third Line.

As to — 1, ſince it gives 0 for third Term, and 0 is found in the third Line, I ſet down in a ſixth Column the Progreſſion — 2, — 1, 0, likewiſe + 1 taken for ſecond Term, giving + 4 which is alſo found in the third Line, I ſet down the ſecond Progreſſion — 2, + 1, + 4, and as + 7 and + 21 ſhould give each a third Term, which is not found in the third Line, I reject them. Having determined the Progreſſions that can begin by 2, I paſs to thoſe whoſe firſt Term will be 0, and to find them I take, as I did for 2, all the Numbers of the ſecond Line one after the other.

I perceive firſt, that — 21 ſhould give for third Term — 42, which is not found in the third Line; I next perceive that — 7 gives — 14 which is found there, I therefore ſet down the Progreſſion 0, — 7, — 14; likewiſe — 3 and — 1, giving — 6 and — 2, which are alſo found in the third Line; I ſet down the Progreſſions 0, — 3, — 6, and 0, — 1, — 2. As to the Numbers + 1, + 7, + 21, becauſe the third Terms they give are not found in the third Line, I reject them.

To diſcover now which of thoſe Progreſſions are to be rejected, I put $x = -2$, and ſet down — 2 in the firſt Column; obſerving at the ſame Time to place, 1°. In the ſecond Column, 125 Number into which the propoſed Quantity is transformed, by ſubſtituting in it this Value of $x$. 2°. In the third, the Numbers 1, 5, 25, 125, Diviſors of 125. 3°. In the fourth Column, the Number 4 ſquare of — 2, and Value of $xx$ in this Suppoſition. 4°. In the fifth Column, the Numbers — 129, — 29, — 9, — 5, found by deducting 4 from the Numbers 125, 25, 5, 1, taken with the Sign —, and the Numbers — 3, + 1, + 21, + 121, found by deducting 4 from the ſame Numbers 1, 5, 25, 125, taken with the Sign +.

By this Means, I perceive that the two Progreſſions — 2, + 1, + 4, and 0, — 7, — 14, are to be rejected, becauſe they ſhould give for fourth Term + 7 and — 21 which are not found in the fourth Line.

But the Progreſſions — 2, — 1, 0; 0, — 3, — 6; 0, — 1, — 2, giving for fourth Terms + 1, — 9, — 3, which are found in this fourth Line, to exclude at leaſt one of thoſe Progreſſions I have need of a new Suppoſition. I therefore put $x = -3$, whereby the propoſed Quantity is transformed into 147. I therefore ſet down 147 in the ſecond Column, and beſide it, in the third, its Diviſors 1, 3, 7, 21, 49, 147; I likewiſe ſet down in the fourth Column 9 Square of — 3, or Value of $xx$ in the new Suppoſition; laſtly, I ſet down in the fifth Column the Numbers — 156, — 58, — 30, — 16, — 12, — 10, — 8, — 6, — 2, + 12, + 40, + 138, found by deducting 9 from the Numbers 1, 3, 7, 21, 49, 147, taken with the Sign — and with the Sign +.

By this Means, I find that the Progreſſions — 2, — 1, 0, + 1; 0, — 1, — 2, — 3, ſhould be rejected, but that the Progreſſion 0, — 3, — 6, — 9, ſhould be retained; becauſe the two fifth Terms of the

first Progressions should be + 2 and — 4, which are not found in the fifth Line, whilst the fifth Term of the Progression 0, — 3, — 6, — 9 is — 12 which is found there.

After having reduced all the Progressions to one, viz. 0, — 3, — 6, &c. I take in this Progression the Number — 3, which, in the sixth Column corresponds to the Supposition of $x = 0$, to express the Term $c$ of the Divisor sought $xx + bx + c$. I afterwards take in the sixth Column likewise, 0 which corresponds to the Supposition of $x = 1$, and which according to the foregoing Principles should express $b + c$; whence deducting $c$ or — 3 from 0, or from $b + c$, I find + 3 for $b$, and consequently the Divisor sought $x^2 + bx + c$ is $xx + 3x - 3$, if the proposed Quantity has one of two Dimensions; I therefore try the Division of this Quantity $x^5 + 3x^4 + 2x^3 + 8x^2 - 36x + 21$ by $xx + 3x - 3$, which succeeds, and gives $x^3 + 5x - 7$ for Quotient.

XXXVIII.

Let it be proposed to find a Divisor of two Dimensions of the Quantity

$$x^4 + 6x^3 + 11x^2 + 10x + 5.$$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 2 | 133 | 1.7.19.133 | 4 | —137, —23, —11, —5, —3, +3, +15, +129 | +15 | +3 |
| 1 | 33 | 1.3.11. 33 | 1 | —34, — 12, — 4, — 2, 0, + 2, + 10, + 32 | +10 | +2 |
| 0 | 5 | 1.5 | 0 | —5, — 1, + 1, + 5 | + 5 | +1 |
| —1 | 1 | 1 | 1 | —2, 0 | 0 | 0 |
| —2 | 3 | 1.3 | 4 | —7, — 5, — 3, — 1 | — 5 | —1 |

Application of the foregoing method to another example.

I set down in a first vertical Column the Values 2, 1, 0, — 1, — 2, which are proposed to be given to $x$; I afterwards set down in the second vertical Column the Numbers 133, 33, 5, 1, 3, into which the proposed Quantity is transformed by those Suppositions; I set down in the third Column opposite to those Numbers all their Divisors. In the fourth the Values 4, 1, 0, 1, 4 of $xx$, resulting from the Suppositions made for $x$ in the first Column; lastly, in the fifth Column I set down, for the first Line, the Numbers — 137, — 23, — 11, — 5, — 3, + 3, + 15, + 129 found by deducting 4 from the Numbers 133, 19, 7, 1, taken first with the Sign — and afterwards with the Sign +. Likewise in the second Line the Numbers — 34, — 12, — 4, — 2, 0, + 2, + 10, + 32 produced by deducting 1 from the Numbers 33, 11, 3, 1, taken first with the Sign — and afterwards with the Sign +. I proceed in like Manner to find the other Lines. All those Numbers being set down, I take in the fifth Line of the fifth Column, the first Number — 7 for first Term of a Progression, and taking at the same Time — 2 in the fourth Line for second Term, I perceive that the third should be + 3, and is not found in the third Line; I therefore pass to 0, which by assuming — 7 for first Term, will give + 7 for third Term, and as + 7 is not found in the third Line,

I conclude that among the Numbers in the fifth Column, there is no Progreſſion that can begin by — 7.

I therefore take — 5 for firſt Term, and I perceive that aſſuming — 2 for ſecond Term, the third will be + 1 which is found in the third Line, but gives for fourth Term + 4, which is not found in the ſecond Line; I therefore reject — 2, and aſſume 0 for ſecond Term, and there reſults + 5, + 10, + 15, for third, fourth, and fifth Terms; and as all theſe Numbers are found in the third, ſecond, and firſt Lines, I ſet down in the ſixth Column the Numbers 15, 10, 5, 0, — 5.

Afterwards taking — 3 for firſt Term, I perceive that neither — 2 nor 0 can be aſſumed for ſecond Terms, becauſe the firſt would produce the Progreſſions — 3, — 2, — 1, 0, + 1, the laſt Term of which is not found in the firſt Line; and the ſecond would produce the Progreſſion — 3, 0, + 3, &c. which fails counting from the third Term + 3 which is not found in the third Line.

It remains to take — 1 for firſt Term, I firſt give it — 2 for ſecond Term which does not ſucceed, but aſſuming 0 for ſecond Term, there reſult for third, fourth, and fifth Terms, the Numbers + 1, + 2, + 3 which are in the third, ſecond, and firſt Lines; I therefore ſet down in ſixth Column the Numbers + 3, + 2, + 1, 0, — 1.

This being done, I perceive that there is no further Occaſion of giving $x$ new Values to exclude one of the two Progreſſions, becauſe the given Quantity being of four Dimenſions, muſt neceſſarily have two Diviſors of two Dimenſions, or none at all; for when a Quantity of four Dimenſions has been found to have a Diviſor of two Dimenſions, the Quotient which is likewiſe a Diviſor, will be alſo of two Dimenſions.

According to this Obſervation, I take indifferently either of the two foregoing Progreſſions, the firſt, for example, in which 5 correſponding to the Suppoſition of $x = 0$, and 10 to the Suppoſition of $x = 1$, there reſults $c = 5$ and $b + c = 10$, that is, $b = 5$. Whence the Diviſor furniſhed by this Progreſſion is $xx + 5x + 5$. I divide therefore the propoſed Quantity by $xx + 5x + 5$, and the Diviſion ſucceeding, gives for Quotient $xx + x + 1$, which is the Diviſor that the other Progreſſion would have furniſhed.

### XXXIX.

If it be propoſed to find the Diviſors of an Equation, ſuch as $6y^4 - y^3 - 21y^2 + 3y + 20 = 0$, whoſe firſt Term has a Coeficient different from Unity, they may be obtained by employing the foregoing Principles without being at the Trouble of transforming it into one that ſhall have the Coeficient of the firſt term Unit, as may be effected by the Method of Art. XXII.

Method of finding the diviſors of one dimenſion when the higheſt term of the equation

To make this appear, we ſhall firſt examine what concerns Diviſors of one Dimenſion. Let $mx + a$ repreſent the ſimple Diviſor of any pro-

has a coefficient different from unity.

poſed Equation. It is manifeſt, that if $x$ be put ſucceſſively equal to 2, 1, 0, — 1, — 2, &c. This Diviſor will beocme in all thoſe Caſes $2m + a$, $m + a$, $a$, $-m + a$, $-2m - a$, &c. Quantities that are in arithmetical Progreſſion in which it is eaſy to obſerve,

1°. That the Difference $m$ of all the Terms is the Coeficient of $x$ in the Diviſor.

2°. That this ſame Difference $m$ is a Diviſor of the Coeficient of the firſt Term of the propoſed Quantity.

3°. That the Term $a$ correſponding to the Suppoſition of $x = 0$, is the Part of the Diviſor clear of $x$.

4°. That the ſame Quantities in arithmetical Progreſſion will be Diviſors of the given Quantity, in which $x$ has been ſucceſſively put equal to 2, 1, 0, — 1, — 2, &c.

From which it follows, that in Order to diſcover ſuch a ſimple Diviſor when there is any, the ſame Proceſs is to be purſued as above for the three firſt Columns. Then among all thoſe Numbers a Progreſſion is to be looked out for, whoſe common Difference is a Diviſor of the Coeficient of the firſt Term of the propoſed Quantity. Laſtly, to employ this Progreſſion, in the Room of $a$ in $mx + a$ the Term of the Progreſſion correſponding to the Suppoſition of $x = 0$ is to be ſubſtituted, and in the Room of $m$ the Number found by deducting any Term of the Progreſſion from that immediately above it.

XL.

Let it be required to find, for Example, the Diviſors of the Quantity

$$6x^4 - x^3 - 21x^2 + 3x + 20$$

| | | | |
|---|---|---|---|
| 2 | 30 | 1 . 2 . 3 . 5 . 6 . 10 . 15 . 30 | 10 |
| 1 | 7 | 1 . 7 | 7 |
| 0 | 20 | 1 . 2 . 4 . 5 . 10 . 20 | 4 |
| — 1 | 3 | 1 . 3 | 1 |
| — 2 | 34 | 1 . 2 . 17 . 34 | — 2 |

I range as uſual in the firſt Column the Suppoſitions 2, 1, 0, — 1, — 2 to be made for $x$, in the ſecond, the Numbers 30, 7, 20, 3, 34 into which the propoſed Quantity is transformed by thoſe Suppoſitions. Laſtly, in the third all the Diviſors of thoſe Numbers.

This being done, in order to diſcover among all the Numbers of the third Column ſome Progreſſion that will ſerve to find the Diviſor ſought; I try the firſt Number 1 of the firſt Line, and I perceive that aſſuming for ſecond Term the firſt Number 1 of the ſecond Line, the Progreſſion 1, 1, 1, &c. that refults cannot be admitted, becauſe it cannot repreſent the Diviſor ſought $mx + a$ which ſhould neceſſarily vary according to the different Values given to $x$. I perceive likewiſe, that 7 cannot be

aſſumed for ſecond Term, becauſe the third Term produced by this Suppoſition would be 13, which is not found in the third Line.

Taking afterwards 1 with the Sign —, I perceive that the 1 of the ſecond Line cannot be aſſumed for ſecond Term, becauſe the third would be 3, which is not found in the third Line. As to 7, it is uſeleſs to try whether the third Term it gives is found in the third Line, ſince the Difference of — 1, to 7 is 8, which is not a Diviſor of the Coeficient of the firſt Term of the propoſed Quantity. Wherefore 1 whether taken with the Sign +, or with the Sign —, is to be rejected.

Taking in like Manner all the other Numbers of the firſt Line, I find only 10 that can have the required Conditions. Aſſuming 7 for ſecond Term, it gives the Progreſſion 10, 7, 4, 1, — 2 whoſe common Difference is 3, Diviſor of the Coeficient of $6x^3$.

Setting down therefore this Progreſſion in the fourth Column, I take the Term + 4 correſponding to the Suppoſition of $x = 0$ to expreſs the Part $a$ of the Diviſor ſought, and deducting the ſame Term + 4 from the Term above it + 7 which repreſents $m + a$, I find 3 for the Expreſſion of $m$, that is, the Diviſor ſought if there is one, muſt be $3x + 4$. I therefore try the Diviſion by this Quantity, which ſucceeds, and gives $2x^3 - 3x^2 - 3x + 5$ for Quotient.

In like Manner, if it was required to find the Diviſors of the Quantity

$24x^5 - 50x + 49x^3 - 140x^2 + 64x + 30$

| | | | | | |
|---|---|---|---|---|---|
| 2 | 42 | 1 . 2 . 3 . 6 . 7 . 14 . 21 . 42 | + 3 | + 3 | + 7 |
| 1 | 3 | 1 . 23 | + 1 | — 1 | + 1 |
| 0 | 30 | 1 . 2 . 3 . 5 . 6 . 10 . 15 . 30 | — 1 | — 5 | — 5 |
| —1 | 297 | 1 . 3 . 9 . 11 . 27 . 33 . 99 . 297 | — 3 | — 9 | —11 |

Here the Diviſors to be tried are $2x - 1$, $4x - 5$ and $6x - 5$; of which $6x - 5$ ſucceeds, there coming out $4x^4 - 5x^3 + 4x^2 - 20x - 6$.

XLI.

To ſhew the Uſe of the foregoing Rules in the Solution of Problems, here follow ſome Examples: Problem ſolved by the foregoing rules.

*A Maſon had 2 cubical Pieces of Marble, the Side of the one exceeded the Side of the other by 7 Inches, and the ſolid Inches in both made* 873; *what were their Dimenſions?*

If the Side of the leſſer Cube be expreſſed by $x$, then the Side of the greater will be $x + 7$; and $x^3 + (x + 7)^3 = 873$ by Queſt. Hence $2x^3 + 21x^2 + 147x - 530 = 0$: Now, by the foregoing Method I find the Progreſſion 7, 5, 3, whoſe common Difference is 2 a Diviſor of the Coeficient of $2x^3$, and taking the Term 5 correſponding to the Suppoſition of $x = 0$, to expreſs $a$ and 2 for $m$, I find that $2x^3 + 21x^2 + 147x - 530$ is diviſible by $2x - 5$, giving for Quotient $x^2 + 13x + 106$, whence $2x = 5$, and $x = \frac{5}{2}$.

XLII.

Another problem solved by the foregoing rules.

*There are two Numbers, the Difference of whose Cubes is 604, and if their Difference be multiplied by the greater, the Product will be 36: What are those Numbers?*

If $x$ = the greater and $y$ the leſſer of theſe Numbers; conſequently $(x-y)\times x = 36$, or $x-y = \frac{36}{x}$. Alſo $x^3 - y^3 = 604$, by Queſt. and dividing the ſecond Equation by the firſt, $x^2 + xy + y^2 = \frac{604x}{36} = \frac{151x}{9}$, alſo by ſecond $x^2 - 2xy + y^2 = \frac{36\times 36}{xx} = \frac{1296x}{xx}$: The Difference of the two laſt is $3xy = \frac{151x}{9} - \frac{1296}{xx}$, therefore $\frac{11664}{151-27y} - y^3 = 604$, whence $27y^4 - 151y^3 + 16308y - 79540 = 0$. Now by the foregoing Method there will be found the Progreſſion 7, 6, 5, 4, 3, whoſe common Difference is 1 a Diviſor of 27 the Coeficient of $y^4$ the higheſt Power of $y$, and taking the Term 5 correſponding to the Suppoſition of $y = 0$ to expreſs $a$, and 1 for $m$, I find that $27y^4 - 151y^3 + 16308y - 79540 = 0$ is diviſible by $y - 5$, giving for Quotient $27y^3 - 16y^2 - 80y + 15908$; Wherefore $y = 5$.

XLIII.

Method of finding the diviſors of two dimenſions when $x$ has a coeficient.

We ſhall now procceed to explain how Diviſors of two Dimenſions, when the higheſt Power of $x$ has a Coeficient, are inveſtigated. Let $mx^2 + bx + c$ repreſent the Diviſor ſought of a given Quantity, it is manifeſt, that the laſt Term $c$ will be a Diviſor of the laſt Term of the given Quantity, and that $m$ will be a Diviſor of the Coeficient of the higheſt Power of $x$ in the ſame Quantity.

Aſſuming therefore one of the Diviſors of the Coeficient of the firſt Term of the propoſed Quantity to expreſs $m$, and performing the ſame Operation as in Art. XXXVI. with this difference, that deducting inſtead of the Squares 16, 9, 4, &c. the Product of theſe Squares into the Number taken for $m$, in like Manner $b$ and $c$ will be found.

If the Diviſor that reſults, by ſubſtituting in $mx^2 + bx + c$, for $m$ the aſſumed Number, and for $b$ and $c$ thoſe that have been determined in conſequence of this Aſſumption, does not ſucceed, another of the Diviſors of the Coeficient of the firſt Term of the given Quantity is to be aſſumed to expreſs $m$, and the Calculation compleated as before. If after having tried all the Diviſors of the Coeficient of the firſt Term, no Diviſor is found by this Method, you may conclude that the propoſed Quantity has no commenſurable Diviſor.

XLIV.

Suppoſe, for Example, you are to find a Diviſor of two Dimenſions of the Quantity

$$4x^5 + 16x^4 - 22x^3 - 14x^2 - 56x + 77$$

| | | | | |
|---|---|---|---|---|
| 2 | 117 | 1.3.9.13.39.117 | 8 | —125,—47,—21,—17,—11,—9,—7,—5,1,5,31,109 |
| 1 | 5 | 1.5 | 2 | —7,—3,—1,3 |
| 0 | 77 | 1 7.11.77 | 0 | —77,—11,—7,—1,1,7,11,77 |
| —1 | 153 | 1.3.9.17,51.153 | 2 | —155,—53,—19,—11,—5,—3,—1,1,7,15,49,151 |
| —2 | 437 | 1.19.23.437 | 8 | —445,—31,—27,—9,—7,11,15,429 |

| |
|---|
| + 5 |
| — 3 |
| —11 |
| —19 |
| —27 |

Having placed, as uſual, in the firſt Column the Numbers 2, 1, 0, — 1, — 2, &c. to which $x$ is ſuppoſed ſucceſſively equal; in the ſecond, the Numbers 117, 5, 77, 153, 437 into which the propoſed Quantity is transformed by ſubſtituting in it thoſe Values of $x$; in the third, all the Diviſors of the Numbers in the ſecond: I firſt try if it has a Diviſor of two Dimenſions, the Coeficient of whoſe firſt Term is Unit, and not finding any, I aſſume for $m$, that is, for the Coeficient of the firſt Term of the Diviſor, the Number 2 which is one of the Diviſors of the Coeficient 4 of the firſt Term of the given Quantity. Application of this method to an example.

I then ſet down in the fourth Column, inſtead of the Squares of the Numbers of the firſt, the Product of thoſe ſame Squares into 2 ſuppoſed Value of $m$, that is, I ſet down in the fourth Column the Numbers 8, 2, 0, 2, 8. I afterwards deduct theſe Numbers from all thoſe of the third Column taken with the Sign — and with the Sign +, which gives for the firſt Line of the fifth Column — 125, — 47, — 21, — 17, — 11, — 9, — 7, — 5, + 1, + 5, + 31, + 109; for the ſecond Line — 7, — 3, &c.

All thoſe Numbers being ſet down, I ſeek for arithmetical Progreſſions among thoſe Numbers, and I find but the Progreſſion + 5, — 3, — 11, — 19, — 27 which I ſet down in the ſixth Column; now taking — 11 correſponding to the Suppoſition of $x = 0$, to expreſs the Number $c$, and deducting this Number from — 3 which is above it, I find the Remainder 8 for to expreſs $b$. The Diviſor therefore that reſults from the Suppoſition of $m = 2$ is $2x^2 + 8x - 11$; and trying the Diviſion it ſucceeds, giving for Quotient $2x^3 - 7$, and without taking the Trouble of going thro' the Calculation that the Suppoſition of $m = 4$ would require, I perceive that it would not ſucceed, becauſe then the Quotient $2x^3 - 7$ could be reſolved into Factors of two Dimenſions, which is impoſſible. Whence the propoſed Quantity has no other Diviſor of two Dimenſions but $2x^2 + 8x - 11$. Every quantity of leſs than ſix dimenſions that has diviſors muſt have ſome of a leſs number of dimenſions than three.

When the Quantity whoſe Diviſors are to be inveſtigated does not exceed the fifth Degree, they may be found by the foregoing Methods; for if this Quantity is found by thoſe Methods to have no ſimple or quadratick Diviſor, we may alſo conclude that it has no cubick Diviſor.

XLV.

If the quantity has six or more dimensions it may have only divisors of three or more dimensions.

But if the Quantity rises to six or more Dimensions, it may possibly not be resolvable but into Factors of more Dimensions than two, the Method to be pursued for finding those Divisors is founded upon the same Principles as the preceding ones.

Thus if it be a cubick Divisor that is required to be found, let it be $mx^3 - nx^2 + rx - s$, by supposing $x$ equal to the Terms of the arithmetical Progression, it will be as follows:

| | | | | | |
|---|---|---|---|---|---|
| $x=$ 3 | $27m - 9n + 3r - s$ | $27m$ | $9n - 3r + s$ | $5n - r$ | $2n$ |
| $x=$ 2 | $8m - 4n + 2r - s$ | $8m$ | $4n - 2r + s$ | $3n - r$ | $2n$ |
| $x=$ 1 | $m - n + r - s$ | $m$ | $n - r + s$ | $n - r$ | $2n$ |
| $x=$ 0 | $-s$ | 0 | $+s$ | $-n - r$ | |
| $x=-1$ | $-m - n - r - s$ | $-m$ | $n + r + s$ | | |

Method for finding divisors of three or more dimensions.

Where the first Differences are not themselves in arithmetical Progression, as in the last Case, but the Differences of its Terms, or the second Differences, are in arithmetical Progression, the common Difference being $2n$, whence $n$ is known. The Quantity $r$ is found in the Column of the second Differences, and $s$ is always to be assumed some Divisor of the last Term of the proposed Equation, as $m$ is of the Coeficient of the first Term; whence all the Coeficients of a Divisor $mx^3 - nx^2 + rx - s$, with which Trial is to be made, may be determined.

If it is a Divisor of four Dimensions that is required, by proceeding in like Manner, you will obtain a Series of Differences whose second Differences are in arithmetical Progression. If it is a Divisor of five Dimensions that is required, you will obtain in the same Manner a Progression whose third Differences will be in arithmetical Progression, and by observing these Progressions, you may discover Rules for determining the Coeficients of the Divisor required.

XLVI.

Hitherto we have only shewn how to find the Divisors of Numeral Equations, but as literal Equations may also have commensurable Divisors, we shall proceed to explain how they are to be investigated.

Suppose, 1°. That the Equation includes only one known Letter with $x$, and that this Equation, to use the Language of the Analysts, is homogeneous, that is, has all its Terms raised to the same Dimension, such for Example, as the Equation $x^3 - ax^2 - 10a^2x + 6a^3$. It suffices to substitute Unity in the Room of the given Letter $a$ of this Equation, and find its Divisors by the preceding Rules. These Divisors being found, if they are of one Dimension, the Letter $a$ is to be substituted again beside the Number which serves for second Term. If the Divisor

is of two Dimenſions, $a$ is to be placed after the Coeficient of the ſecond Term, and $a\,a$ after the Number that ſerves for third Term.

Let the Quantity $x^3 + 4\,a\,x^2 - 17\,a^2\,x - 12\,a^3$ be propoſed, by putting $a = 1$, that Quantity becomes $x^3 + 4\,x^2 - 17\,x - 12$ whoſe Diviſor is $x - 3$, whence $x - 3\,a$ is a Diviſor of the propoſed Quantity.

In like Manner if $2\,x^5 + 5\,a\,x^4 - 3\,a^2\,x^3 - 8\,a^3\,x^2 - 20\,a^4\,x + 12\,a^5$ was propoſed, by putting $a = 1$, it will become $2\,x^5 + 5\,x^4 - 3\,x^3 - 8\,x^2 - 20\,x + 12$, whoſe Diviſor of two Dimenſions is found to be $2\,x^2 + 5\,x - 3$. Subſtituting again in this Diviſor $a$ beſide 5, and $a\,a$ beſide 3, there reſults $2\,x^2 + 5\,a\,x - 3\,a\,a$ for the Diviſor of two Dimenſions of $2\,x^5 + 5\,a\,x^4 - 3\,a^2\,x^3 - 8\,a^3\,x^2 - 20\,a^4\,x + 12\,a^5$.

XLVI.

Method of finding all the diviſors involving two letters in a quantity involving three.

The ſame Rules may ſerve for diſcovering the Diviſors in an homogeneous Equation involving three Letters, but by Help of ſome Obſervations of Calculation which naturally occur, they may be found after a much eaſier Manner.

Suppoſe, 1°. That the Quantity given which involves three Letters $a$, $b$, $x$, ſhould have for Diviſor a Quantity that includes only two Letters $x$ and $a$, for Example. Since this Diviſor, be it what it will, tho' it does not contain $b$, divides the propoſed Quantity in which $b$ enters, the Value of the Letter $b$ muſt not affect the Diviſion, and conſequently this Diviſion may be performed even when $b = 0$. Wherefore if $b$ be put $= 0$ in the given Quantity, the Quantity ariſing from this Suppoſition muſt have for common Diviſor with the whole Quantity, the Diviſor ſought, whence by the Rules of Art. LXXIII. Chap. I. the Diviſor ſought of whatever Number of Dimenſions it, be provided it contains but two Letters, may be found.

XLVII.

To ſhew the Application of this Method, let, 1°. The Quantity $x^4 + a\,x^3 + 2\,a^2\,x^2 + 3\,a^3\,x + a\,b\,b\,x + a^4 + a\,a\,b\,b$ be propoſed, of which it is required to find a Diviſor involving only the Letters $a$, $x$.

Example.

By putting $b = 0$, there reſults $x^4 + a\,x^3 + 2\,a^2\,x^2 + 3\,a^3\,x + a^4$ whoſe common Diviſor with the entire Quantity, or which is the ſame, with the Remainder $a\,b\,b\,x + a\,a\,b\,b$ is $x + a$, which is therefore neceſſarily a Diviſor of the given Quantity, and the greateſt it can admit of that does not contain $b$.

XLVIII.

Another Example.

Let $x^5 - 4\,a\,x^4 + 6\,a\,a\,x^3 - a\,b\,x^3 + a\,b\,b\,x^2 + 2\,a\,a\,b\,x^2 - 4\,a^3\,x^2 - 2\,a\,a\,b\,b\,x - 2\,a^3\,b\,x + 2\,a^3\,b\,b$, be propoſed; by putting $b = 0$ in this Quantity, there reſults $x^5 - 4\,a\,x^4 + 6\,a\,a\,x^3 - 4\,a^3\,x^2$ whoſe greateſt common Diviſor with the propoſed Quantity, or which comes to the ſame, with the Remainder $- a\,b\,x^3 + a\,b\,b\,x^2 + 2\,a\,a\,b\,x^2$

$-2aabbx-2a^3bx+2a^3bb$, that is, the greateſt common Diviſor of the Quantities $x^3-4ax^2+6aax-4a^3$, and $-x^3+bx^2+2ax^2-2abx-2aax+2aab$. Now, if thoſe two Quantities have a common Diviſor that does not contain $b$, it will alſo be common to the two Parts $-x^3+2ax^2-2a^2x$ and $bx^2-2abx+2a^2b$ of the laſt of thoſe two Quantities; but the common Diviſor of thoſe two Parts can only be $xx-2ax+2a^2$, I try therefore if it divides alſo $x^3-4ax^2+6aax-4a^3$, and as it divides it in effect, I conclude that it is the Diviſor ſought of the propoſed Quantity.

XLIX.

Method of finding the diviſors of one dimenſion involving three letters.

Suppoſe now that the propoſed Quantity, involving three Letters whoſe Diviſors are required has no Diviſor, compoſed only of two Letters, or if it has, that they have been found and ſet apart. To find afterwards the Diviſors involving three Letters and of one Dimenſion, that it may contain, I expreſs this Diviſor by $mx+na+pb$, $m$, $n$, $p$ denoting Numbers, I now obſerve that if $a$, $x$, $b$ be put ſucceſſively $=0$ in this Diviſor, there will reſult the three Quantities $mx+pb$, $na+pb$, $mx+na$, ſuch, that the two Terms that each of them contain is found repeated in the two other Quantities. $mx+pb$, for Example, given by the Suppoſition of $a=0$, is compoſed of $mx$ which is found in $mx+na$ ariſing from the Suppoſition of $b=0$, and of $pb$ which is found in $na+pb$ ariſing from the Suppoſition of $x=0$; I perceive at the ſame Time, that the Sum of theſe three Quantities $mx+na$, $mx+pb$, $na+pb$ is the double of the Diviſor $mx+na+pb$.

Now, as thoſe three Quantities are neceſſarliy Diviſors of thoſe to which the propoſed Quantity is reduced by the ſame Suppoſitions of $a$, $x$, $b$ equal 0, it follows that to find the Diviſors of this Quantity which are of one Dimenſion and involve three Letters, we muſt firſt ſet down ſeparately the three Quantities into which the propoſed one is transformed, by the Suppoſition of $a$, $x$, $b=0$, then ſet down beſide each of thoſe new Quantities all its Diviſors involving two Letters of one Dimenſion. This being done, we muſt ſelect three Diviſors out of thoſe three Claſſes of Diviſors involving two Letters, having the Conditions above-mentioned, *viz.* that the two Terms of which each of thoſe Diviſors conſiſts, is found in the other two Diviſors. Thoſe three Diviſors being thus found, the Half of their Sum will be the Diviſor of the propoſed Quantity, if it has one.

If it be found neceſſary in order to have in one of thoſe three Diviſors of two Letters, the two Terms which ſhould be the Repetition of thoſe that are in the two others, to change their Signs, it is eaſy to perceive that this Change is admiſſable, ſince in general a Quantity that divides another, will ſtill divide it, when the Signs of all its Terms are changed.

L.

To ſhew the Application of this Method, let it be propoſed to find the Diviſors involving three Letters of the Quantity

$$2x^3 + 7ax^2 - 3bx^2 + 5a^2x - 3abx + 4b^2x + 10ab^2 - 6b^3$$

| | | |
|---|---|---|
| $10ab^2 - 6b^3$ | $5a - 3x,\ 10a - 6b$ | $5a - 3b$ |
| $2x^3 - 3bx^2 + 4bbx - 6b^3$ | $2x - 3x$ | $2x - 3b$ |
| $2x^3 + 7ax^2 + 5a^2x$ | $2x + 5a$ | $2x + 5a$ |

$5a + 2x - 3b$

Application of the foregoing method to an example.

Having ſet down in a vertical Column the three Quantities $10ab^2 - 6b^3$; $2x^3 - 3bx^2 + 4bbx - 6b^3$; $2x^3 + 7ax^2 + 5a^2x$, into which this Quantity is transformed by the Suppoſition of $x$, $a$, $b$ equal 0; I ſet down in a ſecond vertical Column oppoſite each of theſe three Quantities, their Diviſors of one Dimenſion involving two Letters. The firſt gives $5a - 3b$ and $10a - 6b$; the ſecond only $2x - 3b$, and the third $2x + 5a$. I now perceive that if of the two Diviſors $5a - 3b$, $10a - 6b$, the firſt $5a - 3b$ be taken; it will have with the two Diviſors $2x - 3b$, $2x + 5a$, the Conditions required. For this firſt Diviſor $5a - 3b$ contains $5a$, which is repeated in the Diviſor $2x + 5a$, and $-3b$ which is repeated in $2x - 3b$; likewiſe $2x - 3b$ contains $2x$, which is repeated in $2x + 5a$, and $-3b$ which is repeated in $5a - 3b$; laſtly, $2x + 5a$ is compoſed of $2x$ and of $+5a$, which are repeated in the two others $5a - 3b$, $2x - 3b$. I therefore, according to the foregoing Rule, add theſe three Diviſors together, and there reſults $4x - 6b + 10a$, the one half of which $2x - 3b + 5a$ is the Diviſor ſought. In effect trying the Diviſion, I find for Quotient $x^2 + ax + 2bb$.

LI.

Let it be propoſed to find the Diviſors of one Dimenſion involving three Letters, of the Quantity

$$8x^4 - 2ax^3 - 10bx^3 - 3a^2x^2 - 5abx^2 - 12ab^2x + 9a^2b^2 + 15ab^3$$

| | | |
|---|---|---|
| $9a^2b^2 + 15ab^3$ | $3a + 5b,\ 9a + 15b$ | $-3a - 5b$ |
| $8x^4 - 10bx^3$ | $4x - 5b,\ 8x - 10$ | $4x - 5b$ |
| $8x^4 - 2ax^3 - 3a^2x^2$ | $4x^3 - 3a,\ 2x + a$ | $4x - 5b$ |

$4x - 3a - 5b$

Application of the foregoing method to another example.

Putting in this Quantity $x$, $a$, $b$ ſucceſſively $= 0$, there reſults the three Quantities $9a^2b^2 + 15ab^3$, $8x^4 - 10bx^3$, $8x^4 - 2ax^3 - 3a^2x^2$, whieh I ſet down one under the other in a vertical Column. I ſet down

in another vertical Column beſide each of theſe Quantities, their Diviſors of one Dimenſion involving two Letters; thoſe of the firſt are $3a + 5b$ and $9a + 15b$; thoſe of the ſecond $4x - 5b$, $8x - 10b$; and thoſe of the third $4x - 3a$ and $2x + a$.

It is now eaſy to perceive that the three Diviſors $3a + 5b$, $4x - 5b$, and $4x - 3a$ have the Conditions required, provided the Signs of the firſt be changed, that is, by ſetting it down thus $-3a - 5b$; I therefore ſet apart thoſe three Diviſors in the fourth Column, I add them together, and I take the one half of their Sum, which gives $4x - 3a - 5b$ for the Diviſor ſought, and trying the Diviſion, it ſucceeds, giving for exact Quotient $2x^3 + ax^2 - 3ab^2$.

LII.

The foregoing method applied for finding the diviſors involving three letters at the ſame time as thoſe involving two letters.

In thoſe two Examples the Diviſors of one Letter of each of the three Quantities of the firſt Column were not ſet down, becauſe thoſe Diviſors could never be the Quantities into which the Diviſor involving three Letters is transformed by the Suppoſition of $x$, $a$, $b$ equal 0, and beſides it has been ſuppoſed that it has been found by the Method of Art. XLVI. that the propoſed Quantity has no Diviſor involving two Letters, but if the propoſed Quantity has Diviſors of this Sort, they may be found at the ſame Time as thoſe involving three Letters, by the foregoing Method, provided they be but of one Dimenſion.

Suppoſe, for Example, you are to find the Diviſors of the quantity

$$16x^3 + 16bx^2 - 48ax^2 + 35a^2x - 16abx - 6a^3 + 3a^2b.$$

$-6a^3 + 3aab$
$16x^3 + 16bxx$
$16x^3 - 48axx + 35aax - 6a^3$

| $a, 3a, -2a+b, -6a+3b$
| $\left\{\begin{matrix} x, 2x, 4x, 8x, 16x, x+b, 2x+2b \\ 4x+4b, 8x+8b, 16+16b \end{matrix}\right.$
| $a-4x, 3a-4x, x-2a$

| $a$ | $2a$ | $-2a+b$
| $-4x$ | $-4x$ | $x+b$
| $a-4x$ | $3a-4x$ | $x-2a$

$a-4x \mid 3a-4x \mid x-2a+b$

Having ſet down in a firſt Column the three Quantities $-6a^3 + 3a^2b$, $16x^3 + 16bxx$, $16x^3 - 48ax^2 + 35a^2x - 6a^3$ into which this Quantity is transformed by the Suppoſition of $x$, $a$, $b$ equal 0; I ſet down in the ſecond Column, and in the firſt Line $a$, $3a$, $-2a+b$, $-6a+3b$ Diviſors of one Dimenſion involving one and two Letters of the Quantity $-6a^3 + 3a^2b$, likewiſe I ſet down in the ſecond Line the Diviſors $x$, $2x$, $4x$, $8x$, $16x$, $x+b$, $2x+2b$, $4x+4b$, $8x+8b$, $16x+16b$, of the ſecond Quantity $16x^3 + 16bxx$: and in the third Line, $a - 4x$, $3a - 4x$, $x - 2a$ Diviſors of the third Quantity $16x^3 - 48aax + 35a^2x - 6a^3$.

Now on account of the great Number of theſe Diviſors, in order to omit none of them that may have the Conditions required, I obſerve

much the ſame Order as in trying the numerical Diviſors. Comparing the firſt Diviſor of the firſt Line with all thoſe of the other Lines, and performing afterwards the ſame Operation for each of the other Diviſors of the firſt Line, I perceive firſt, That if $a$ is a Part of a Diviſor of the propoſed Quantity, it muſt be a Diviſor containing only $a$ and $x$, becauſe if it contained another Term $b$, this Diviſor could not be reduced to $a$ by the Suppoſition of $x = 0$. In this Caſe therefore we have only to chuſe among the five firſt Diviſors $x$, $2x$, $4x$, $8x$, $16x$. Now, as among all thoſe Diviſors, only $4x$ is repeated in the third (provided that this Diviſor is affected with the Sign —), and at the ſame Time of all the Diviſors of the third Line, there is only the firſt $a - 4x$ which includes the ſame Term $a$ of the firſt Line. I conclude that if $a$ be a Part of a Diviſor, this Diviſor muſt be $a - 4x$; I therefore ſet it down a Part. I afterwards paſs to $3a$, and as I find it repeated in the Diviſor $3a - 4x$ of the third Line, and that the other Term $4x$ of the ſame Diviſor is found among the Diviſors of the ſecond Line, after changing the Sign of this Diviſor; I conclude that $3a - 4x$ may be alſo a Diviſor of the propoſed Quantity, and I ſet it apart likewiſe in order to try it.

As to $-2a + b$ it cannot alone be a Diviſor of the propoſed Quantity, becauſe then among the Diviſors of $16x^3 + 16bxx$ the Term $b$ ſhould be found, to which $-2a + b$ is reduced by the Suppoſition of $a = 0$. It remains therefore to try if it be not a Part of a Diviſor in which $x$ enters; I obſerve that of all the Diviſors of the ſecond Line, only $x + b$ can be compared with it, becauſe it is the only Diviſor that has the Term $b$ in common with it. I perceive likewiſe, that of all the Diviſors of the third Line only $x - 2a$ can be compared with the ſame Diviſor $-2a + b$, becauſe it is the only one that contains the Term $-2a$. I obſerve, laſtly, that as the two Terms of the Diviſor $-2a + b$ are repeated in the two other Diviſors $x + b$, $x - 2a$, ſo likewiſe the two Terms of the Diviſor $x + b$ are repeated in the two others $-2a + b$, $x - 2a$, and reciprocally that the two Terms of the Diviſor $x - 2a$ are repeated in the two others $x + b$, $-2a + b$. Whence I conclude that the three Diviſors $-2a + b$, $x + b$, $x - 2a$, have the required Conditions to form a Diviſor, I therefore add them together, and ſet apart the one Half $x - 2a + b$ of their Sum for a Diviſor to be tried. As to the Diviſor $6a + 3b$, I perceive at once that neither of its two Terms are repeated among the Diviſors of the other Lines, and conſequently it is to be rejected.

Having thus found the three Diviſors $a - 4x$, $3a - 4x$, $x - 2a + b$ with which Trial is to be made, I try the Diviſion by the laſt, which ſucceeds, and gives for Quotient $3aa - 16ax + 16xx$, which I divide afterwards by $3a - 4x$. The Diviſion likewiſe ſucceeds, and gives for

Quotient the first Divisor $a - 4x$, whence the proposed Quantity is the Product of those three Divisors.

LIII.

Method of finding the divisors of two dimensions involving three letters.

If the proposed Quantity has no Divisor of one Dimension, we are to enquire if it has not some quadratic Divisor. Let $mx^2 + nax + pbx + qb^2 + rab + sbb$ expres this Divisor; putting successively $x = 0$, $a = 0$, $b = 0$, in this Divisor, there results $qa^2 + rab + sbb$; $mx^2 + pbx + sbb$; $mx^2 + nax + qa^2$; which are all three Divisors of the Quantities into which the proposed one is transformed by the same Suppositions of $x$, $a$, $b$ equal 0. Moreover, each of those Divisors is such, that the Terms affected with square Letters are repeated in the two other Divisors, whilst the Terms, that contain a Product of two Letters are alone of their Species. When, therefore, among the Divisors of two Dimensions and involving two Letters of the Quantities to which the proposed Quantity is reduced by the foregoing Suppositions, three are found having the above-mentioned Conditions, by adding them together, and taking the one half of all the Terms affected with Squares, and leaving those that are rectangles entire, a Divisor of two Dimensions to be tried will be obtained.

LIV.

Application of this method to an example.

To show the Application of this Method, let it be proposed to discover whether the following Quantity, which has no simple Divisor, has any quadratick Divisors:

$$x^5 - 4ax^4 - 5a^2x^3 + 4b^2x^3 - a^3x^2 - 14ab^2x^2 + 3b^4x - a^4x - 3a^3b^2$$

$$-3a^3b^2 \quad \left|\begin{array}{l} a^2, ab, b^2, 3a^2, 3ab, 3b^2 \\ x^2 + 3b^2, x^2 + b^2 \\ x^2 + ax \end{array}\right. \quad \left|\begin{array}{r} -b^2 \\ x^2 + b^2 \\ x^2 + ax \\ \hline x^2 + b^2 + ax \end{array}\right. \quad \left|\begin{array}{r} 3b^2 \\ x^2 + 3b^2 \\ x^2 + ax \\ \hline x^2 + 3b^2 + ax \end{array}\right.$$

$$x^5 + 4b^2x^3 + 3b^4x$$

$$x^5 - 4ax^4 - 5a^2x^3 - a^3x^2 - a^4x$$

I set down in a first Column the three Quantities $-3a^3b^2$; $x^5 + 4b^2x^3 + 3b^4x$; $x^5 - 4ax^4 - 5a^2x^3 - a^3x^2 - a^4x$; into which the proposed Quantity is transformed by the Suppositions of $x$, $a$, $b$ equal 0. I afterwards set down beside these Quantities their Divisors of two Dimensions; the first gives $a^2$, $ab$, $b^2$, $3a^2$, $3ab$, $3bb$; the second $x^2 + 3b^2$, $x^2 + b^2$; the third only $x^2 + ax$. In this Method, the Divisors consisting only of one Term are not to be rejected, even tho' it has been found that the proposed Quantity has no Divisor involving two Letters, because a Divisor consisting of three Terms and of two Dimensions, may be reduced to one Term, by supposing one of the Letters equal 0.

It remains now to try all the Diviſors of the firſt Line; I perceive, 1°. That the firſt $a^2$ is to be rejected, becauſe this Square is not repeated in the other Lines. I paſs afterwards to $a\,b$, and as this Diviſor does not contain any Term affected with $a\,a$ or with $b\,b$, I conclude that the Diviſor, of which it might be a Part, cannot have, beſides this Term, any other but $x\,x$, or $a\,x$, or $b\,x$, whereby the Compariſon of $a\,b$ with the Diviſors $x^2 + 3\,b^2$, $x^2 + b^2$ being excluded, it follows that if $a\,b$ be a Part of a Diviſor of the propoſed Quantity, this Diviſor muſt be $x\,x + a\,x + b\,a$; but at the ſame Time, I perceive that $x\,x + a\,x + b\,a$ cannot be a Diviſor of the propoſed Quantity, ſince it would be reduced to $xx$ by the Suppoſition of $a = 0$, and that $xx$ is not one of the Diviſors of the ſecond Line. Wherefore the Diviſor $a\,b$ is likewiſe to be rejected.

As to the Diviſor $b\,b$, I find it repeated in the Diviſor $x^2 + b^2$ of the ſecond Line, and finding that the ſame Diviſor $x^2 + b^2$, has $x^2$ in common with the Diviſor of the third Line, I conclude that $x^2 + b^2 + a\,x$ has the Conditions required. I afterwards paſs to the other Diviſors of the firſt Line, and I perceive, 1°. That $3\,a\,a$ and $3\,a\,b$ are to be rejected, in like Manner as $a^2$ and $a\,b$; I obſerve afterwards that $3\,b\,b$ is repeated in the Diviſor $x^2 + 3\,b^2$ and $x^2$ in $x\,x + a\,x$, whence I conclude that $x\,x + 3\,b\,b + a\,x$ has alſo the Conditions required. Trying thoſe two Diviſors, I find that the ſecond only ſucceeds, giving for Quotient $x^3 - 5\,a\,x^2 + b^2\,x - a^3$.

Inſtead of trying all the Diviſors of the firſt Line, we might, by trying that one alone which is contained in the laſt Line, find much more readier that $x^2 + a\,x + b^2$ and $x^2 + a\,x + 3\,b^2$ are the only Diviſors to be tried. For obſerving that the Diviſor $x^2 + a\,x$ contains $x^2$ which is repeated in $x^2 + 3\,b^2$ and in $x^2 + b^2$, and that thoſe two laſt contain; the one $3\,b^2$, and the other $b^2$, which are each found among the Diviſors of the firſt Line, it is eaſy to conclude that $x^2 + a\,x + b^2$ and $x^2 + a\,x + 3\,b^2$ have the required Conditions, and that they are the only ones, ſince if there were others, they would give different Quantities from $x\,x + a\,x$ by ſuppoſing $b = 0$, or different Quantities from $x^2 + 3\,b^2$ and $x^2 + b^2$ by ſuppoſing $a = 0$.

LV.

Caſe in which the diviſors are found much eaſier than by the foregoing method.

If ſome Letter in the propoſed Quantity is only of one Dimenſion, it is eaſy to perceive that only one of the Diviſors of this Quantity can contain it, whence there will be at leaſt one Diviſor that does not contain it, and conſequently to find this Diviſor (Art. XLVI.) you may ſeek for the greateſt common Diviſor of the Terms in which that Letter is found, and of the remaining Terms in which it is not found. Thus, in the Quantity $x4 - 3\,a\,x^3 - 8aaxx + 18\,a^3\,x + c\,x^3 - acxx - 8aacx + 6a^3c - 8a^4$, the common Diviſor of the Terms $cx^3 - acxx - 8aacx + 6a^3c$ and of $x4 - 3ax^3 - 8caxx + 18a^3x - 8a4$, *viz.* $xx + 2ax - 2aa$, will divide the whole quantity.

LVI.

Another example.

Let it now be proposed to find the Divisors of $2x^5 + 3ax^4 + b^2x^3 - a^2x^3 + 4ab^2x^2 + 6a^2b^2x + 2ab^4 - 2a^3b^2$ involving two or three Letters, either of one or two Dimensions,

| | |
|---|---|
| 1 | $2ab^4 - 2a^3b^4$ |
| 2 | $ab^4 - a^3bb$ |
| $2a, a$ | $b^4 - a^2b^2$ |
| $2ab, ab, 2b, b$ | $b^3 - a^2b$ |
| $2ab^2, ab^2, 2b^2, b^2b$ | $b^2 - a^2$ |
| $b^3 - ab^2, 2ab^2 - 2a^2b, ab^2 - a^2b, 2b^2 - 2ab, b^2 - ab, 2ba - 2a^2, ba - a^2, 2b - 2a, b - a$ | $b + a$ |
| $2b^3 - 2ab^2, ab^3 - a^2b^2, 2ab^3 - 2a^2b^2)$ | |
| $b^3 + ab^2, 2ab^2 + 2a^2b, ab^2 + a^2b, 2b^2 + 2ab, b^2 + ab, 2ba + 2a^2, ba + a^2, 2b + 2a, b + a,$ | |
| $2b^3 + 2ab^2, ab^3 + a^2b^2, 2ab^3 + 2a^2b^2)$ | |
| $2ab^3 - 2a^3b, ab^3 - a^3b, 2b^3 - 2a^2b, b^3 - a^2b, 2b^2a - 2a^3, b^2a - a^3, 2b^2 - 2a^2, b - a$ | |
| $b^4 - a^2b^2, 2b^4 - 2a^2b^2, ab^4 - a^3b^2, 2ab^4 - 2a^3b^2)$ | |

$2x^5 + 3ax4 + b^2x^3 - a^2x^3 + 4ab^2x^2 + 6a^2b^2x + 2ab4 - 2a^3b^2$

| | | | | |
|---|---|---|---|---|
| $2ab4 - 2a^3b^2$ | $a, 2a, b, 2b, b - a, 2b - 2a, b + a, 2b + 2a$<br>$bb, 2bb, ab, 2ab, bb - aa, 2bb - 2aa, ba - aa,$<br>$ba + aa, bb - ab, bb + ab, 2ba - 2aa,$<br>$2ba + 2aa, 2bb - 2ab, 2bb + 2ab$ | $ab$ | $2ab$ | $bb - aa$ |
| $2x^5 + b^2x^3$ | $x, xx, 2xx + bb$ | $xx$ | $xx$ | $2xx + bb$ |
| $x^5 + 3ax4 - a^2x^3$ | $x, x^2, 2x^2 + 3ax - aa$ | $xx$ | $xx$ | $2x^2 + 3ax - aa$ |
| | | $x^2 + ab$<br>$x^2 - ab$ | $x^2 + 2ab$<br>$x^2 - 2ab$ | $2x^2 + b^2 + 3ax - a^2$ |

$2ab^4 - 2a^3b^2$, $2x^5 + b^2x^3$, and $2x^5 + 3ax^4 - a^2x^3$ being the Quantities to which the proposed one is reduced by the Supposition of $x$, $a$, $b$ equal 0, opposite each of these Quantities must be ranged all the Divisors they may admit of, as well of one Dimension as of two. As the first of those three Quantities admits of a great Number, in order to omit none of them, I follow the same Order as was observed in the investigation of the numerical Divisors.

Application of the method of Art. xxxv. for finding all the divisors of literal quantities.

Having set down this first Quantity $2ab^4 - 2a^3bb$, and drawn a Line to the left Hand of it, I write down Unity, as being its first Divisor; I afterwards set down 2 under 1, because it is after 1 the simplest Divisor that this Quantity will admit of, and I set down to the right Hand of the same Line $ab^4 - a^3b^2$. I afterwards divide this Quantity by $a$, and set down $a$ to the left Hand of the vertical Line, writing down at the same Time to

the right Hand the Quotient $l^4 - a^2 b^2$. I next multiply $a$ by 2, and ſet down $2a$ to the left Hand of $a$, as being a new Diviſor of the propoſed Quantity; I afterwards divide $l^4 - a^2 b^2$ by $b$, and ſet down the Diviſor $b$ to the left Hand, and the Quotient $b^3 - a^2 b$ to the right. Laſtly, I multiply $b$ by 2, by $a$, by $2a$; and ſet down to the left Hand of $b$, the Products $2b$, $ab$, $2ab$, as being new Diviſors of the propoſed Quantity.

The propoſed Quantity being reduced to $b^3 - a^2 b$, I divide it again by $b$, which I ſet down to the left Hand, and the Quotient $b^2 - a^2$ to the right; I omit multiplying $b$ by 2, or by $a$, or by $2a$, becauſe there would reſult the Diviſors already found; but I multiply it by $b$ and by $2b$, which produces the new Diviſors of two Dimenſions $bb$ and $2bb$; if Diviſors of three Dimenſions were to be admitted, as may be neceſſary on other Occaſions, then $b$ ſhould be likewiſe multiplied by $ab$ and $2ab$.

After having reduced the Quantity to $bb - aa$, I perceive that it is diviſible by $b - a$, and the Quotient is $b + a$. I therefore ſet down one to the left and the other to the right Hand, and I multiply $b - a$ by 2, by $a$, by $2a$, by $b$, and by $2b$, which give for new Diviſors, $2b - 2a$, $ba - aa$, $bb - ab$, $b^2$, $2b^2$, $2ba - 2aa$, $2bb - 2ab$. And if Diviſors of three or four Dimenſions were required, I ſhould have alſo multiplied $b - a$ by $ab$, $2ab$, $abb$, $2abb$. The Quantity remaining $b + a$ admitting of no Diviſor, I ſet it down to the left Hand, and I multiply it by 2, by $a$, by $b$, by $2b$, $2a$, $b - a$, $2b - 2a$, which give for new Diviſors of one and two Dimenſions, $2b + 2a$, $ba + aa$, $2ba + 2aa$, $bb + ab$, $2bb + 2ab$, $bb - aa$, $2bb - 2aa$. If Diviſors of three, four, five Dimenſions, that is, all the Diviſors that the propoſed Quantity can admit of were required, I would have alſo multiplied $b + a$ by $ab$, $2ab$, $bb$, $2bb$, $abb$, $2abb$, $ba - aa$, $2ba - 2aa$, $bb - ab$, $2bb - 2ab$, $abb - aab$; $2abb - 2a^2 b$, $b^3 - ab^2$, $2b^3 - 2ab^2$, $ab^3 - a^2 b^2$, $2ab^3 - 2a^2 b^2$.

This being done, I ſet down all the Diviſors of one and two Dimenſions, $a$, $2a$, $b$, $2b$, $b - a$, $2b - 2a$, $b + a$, $2b + 2a$, $ab$, $2ab$, $ab - aa$, $2ba - 2aa$, $b^2 - ab$, $2b^2 - 2ab$, $ba + aa$, $2ba + 2aa$, $b^2 + ba$, $2bb + 2ab$, beſide the Quantity $2ab^4 - 2a^3 b^2$ which produced them. I afterwards ſet down beſide the Quantity $2x^5 + b^2 x^3$ its Diviſors of one and two Dimenſions $x$, $2xx$, $xx + bb$, and beſide the Quantity $2x^5 + 3ax^4 - a^2 x^3$ its Diviſors of one and two Dimenſions $x$, $x^2$, $2x^2 + 3ax - a^2$.

Now, trying all thoſe Diviſors to diſcover thoſe that are to be rejected, I ſoon perceive that all thoſe of one Dimenſion are of this Sort; for $x$ being the only Quantity of one Dimenſion in the ſecond and third Lines, I conclude that it muſt be the only Diviſor of one Dimenſion that the propoſed Quantity can admit of; ſince if the Diviſor of one Dimenſion

included either a Term affected with $a$, or one affected with $b$, the Term affected with $a$, would remain in the Diviſors given by the Suppoſition of $b = 0$, and the one affected with $b$ would remain in the Diviſors given by the Suppoſition of $a = 0$; but $x$ is not a Diviſor of the propoſed Quantity, conſequently it has no Diviſor of one Dimenſion.

I paſs next to the Diviſors of two Dimenſions, beginning by $x^2$ which I take in the third Line; finding it alſo in the ſecond Line, I conclude that if it makes a Part of a Diviſor, it can only want a Term affected with the Rectangle $a\,b$; in effect if there were Terms in this Diviſor affected with $a\,a$, with $b\,b$, with $a\,x$, or with $b\,x$, thoſe affected with $b\,b$ or with $b\,x$ would not have diſappeared by the Suppoſition of $a = 0$; and thoſe affected with $a\,a$, or with $a\,x$, would not have vaniſhed by the Suppoſition of $b = 0$. But I find in the firſt Line $ab$, $2ab$; wherefore $x\,x + a\,b$, $x\,x + 2\,a\,b$, $x\,x - a\,b$, $x\,x - 2\,a\,b$, are the Diviſors to be tried.

I afterwards paſs to the Diviſor $2x^2 + 3ax - aa$, and I find the Term $2x^2$ repeated in the Diviſor $2\,x\,x + b\,b$ above it, as likewiſe the Term $-aa$ repeated in ſeveral Diviſors that are in the upper Lines; but of all the Diviſors in which it is repeated there is only $bb - aa$, which has the Term $bb$ common with the Diviſor $2xx + bb$. Whence it is only with $2xx + bb$, and $bb - aa$, that $2x^2 + 3ax - aa$ can concur to form a Diviſor that will have the Conditions required, and this Diviſor which is $2x^2 + 3ax - aa + bb$ is conſequently to be tried.

I perceive that it is uſeleſs to ſeek for another Diviſor, becauſe if there could be found one that could not be diſcovered by trying the Diviſors of the third Line, it would conſiſt of one Term affected with $a\,b$: Now it is eaſy to perceive that the propoſed Quantity admits of no Diviſor of this kind. I therefore attempt the Diviſion by $2x^2 + 3ax - a^2 + bb$, which ſucceds, and gives for Quotient $x^3 + 2\,a\,b\,b$, whereby it appears that none of the Quantities $x\,x + a\,b$, $x\,x - a\,b$, $x\,x + 2\,a\,b$, $x\,x - 2\,a\,b$ can divide the propoſed one.

LVII.

If the propoſed Quantity is of more than five Dimenſions, and having found that it has no Diviſor of one or of two Dimenſions, then we are to enquire by a ſimilar Proceſs if it has not ſome Diviſor of three or four Dimenſions.

LVIII.

How the diviſors of quantities that are not homogeneous are found.

When the propoſed Quantity is not homogeneous, that is, when all its Terms are not of the ſame Dimenſions, it is to be rendered homogeneous by means of a new Letter, and the Diviſors of this new Quantity being found by the preceding Rules, the new Letter introduced is to be exterminated by putting it equal to Unit, and by this Means the Diviſors required of the propoſed Quantity will be obtained. Suppoſe, for Example,

the Quantity to be $y^6 + by^5 - by^4 + y^2 + by - b$; I multiply the Term $by^4$ by $a$, to make it of six Dimensions. For the same Reason I multiply $y^2$ by $a^4$, $by$ by $a^4$, and $b$ by $a^5$, whence there results the Quantity $y^6 + by^5 - aby^4 + a^4yy + a^4by - a^5b$, which by the foregoing Method is found to be the Product of $yy - ab + by$ into $y^4 + a^4$. I suppose $a = 1$ in those Factors, and there results $yy - b + by$, and $y + 1$ for the two Factors of the proposed Quantity $y^6 + by^5 - by^4 + y^2 + by - b$.

LIX.

To find the incommensurable Roots of Equations of every Degree, 1°. Substitute successively in the proposed Equation 1, 2, 3, 4, &c. in place of the unknown Quantity, until the Quantity that results after each Substitution changes its Sign, then the Root $y$ will be a positive Number greater than that which immediately precedes the Change of the Sign of the Result, and less than that where this Change happens.

Method of approximating to the roots of numeral equations involving one unknown quantity.

2°. Instead of putting the first Member of the Equation equal 0, put it $= z$; then find the limiting Ratio of the cotemporary Increments of $y$ and $z$, and let it be expressed by A. Substitute the Value of $y$, estimated pretty near the Truth by the foregoing Method, in the Equation, as also in the Value of A, and let the Error or resulting Number in the former, be divided by this numerical Value of A, and the Quotient be subtracted from the said former Value of $y$, and from thence will arise a new Value of that Quantity much nearer to the truth than the former, wherewith proceeding as before, another new value may be had, and so another, &c. till we arrive to any Degree of Accuracy desired.

LX.

Let it be proposed to find a Value of $y$ in the Equation $y^4 + 2y^3 - 36yy + 5y - 116 = 0$.

Application of the foregoing method to an Example.

$$\left.\begin{array}{l} \text{If } y = 1 \\ \text{If } y = 2 \\ \text{If } y = 3 \\ \text{If } y = 4 \\ \text{If } y = 5 \\ \text{If } y = 6 \end{array}\right\} \text{ then } \left\{\begin{array}{rrrrrl} 1 + & 2 - & 36 + & 5 - & 116 = & -144 \\ 16 + & 16 - & 144 + & 10 - & 116 = & -218 \\ 81 + & 54 - & 324 + & 15 - & 116 = & -290 \\ 256 + & 128 - & 576 + & 20 - & 116 = & -288 \\ 625 + & 50 - & 900 + & 25 - & 116 = & -116 \\ 1296 + & 432 - & 1296 + & 30 - & 116 = & +346 \end{array}\right.$$

Whence I conclude that one of the Values of $y$ will be greater than 5, but less than 6. Let it be supposed to be 5, 5, to find a more approximate Value of it, I investigate the limiting Ratio of the cotemporary Increments $dy$ and $dz$ of $y$ and $z$ in the Equation $y^4 + 2y^3 - 36yy + 5y - 116 = z$, which I find to be $4y^3 + 6yy - 72y + 5$, then because $y - \frac{y^4 + 2y^3 - 36yy + 5y - 116}{4y^3 + 6yy - 72y + 5} = y$. I substitute in the first Member of this Equation, in Place of $y$, its Value 5,5 already found, whence we have $5{,}5 - \frac{70{,}3125}{456} = 5{,}340$ for a new Value of $y$, with which pro-

ceeding as before the next Value of $y$ will be found to be 5,3354, and from thence the third Value of $y = 5{,}335438$, which is true to the last Decimal Place.

LXI.

Application of the foregoing method to another example.

Let $\sqrt{(1-y)} + \sqrt{(1-2yy)} + \sqrt{(1-3y^3)} - 2 = 0$, be proposed. one of the Values of $y$ in this Equation will be nearly 0,5. To find a more approximate Value of it, I investigate the limiting Ratio of the cotemporary Increments $dy$ and $dz$ of $y$ and $z$ in the Equation $\sqrt{(1-y)} + \sqrt{(1-2yy)} + \sqrt{(1-3y^3)} - 2 = z$, which I find to be $-\frac{1}{2\sqrt{(1-y)}} - \frac{2y}{\sqrt{(1-2yy)}} - \frac{9yy}{2\sqrt{(1-3y^3)}}$; whence we have $0{,}5 + \frac{0.204}{3.545} = 0{,}557$ for the next Value of $y$, from whence by proceeding as before, the next following will be found 0.5516, &c.

LXII.

When there are two equations given, and as many Quantities $x$ and $y$ how to determine those Quantities.

When there are two Equations given, and as many Quantities ($x$ and $y$) to be determined, find the Equation expressing the limiting Ratio of the cotemporary Increments $dx$ and $dy$ of $x$ and $y$ in both Equations, and in the former collect all the Terms, affected with $dx$, under their proper Signs, and having divided by $dx$ put the Quotient $= A$, and let the remaining Terms, divided by $dy$ be represented by $B$: in like Manner, having divided the Terms in the latter affected with $dx$, by $dx$, let the Quotient be put $= a$, and the Rest divided by $dy$, $= b$. Assume the Values of $x$ and $y$ pretty near the Truth, and substitute in both the Equations, marking the Error in each, and let these Errors, whether positive or negative, be signified by $A$ and $r$ respactively, substitute likewise in the Values of $A, B, a, b$, and let $\frac{Br-bR}{Ab-aB}$ and $\frac{aR-Ar}{Ab-aB}$ be converted into Numbers, and respectively added to the former Values of $x$ and $y$, and thereby new Values of those Quantities will be obtained; from whence, by repeating the Operation, the true Values may be approximated *ad Libitum*.

LXIII.

Application of the foregoing method to an example.

Let there be given the Equations $y + \sqrt{(yy-xx)} - 10 = 0$, and $x + \sqrt{(yy+x)} - 12 = 0$; to find $x$ and $y$.

The Equations expressing the Relation of the cotemporary Increments $dx$, and $dy$, of $x$, and $y$, being here $dy + \frac{y\,dy - x\,dx}{\sqrt{(yy-xx)}} = 0$, and $dx + \frac{y\,dy + \frac{1}{2}dx}{\sqrt{(yy+x)}} = 0$, or $dy + \frac{y\,dy}{\sqrt{(yy-xx)}} - \frac{x\,dx}{\sqrt{(yy-xx)}} = 0$, and $dx + \frac{\frac{1}{2}dx}{\sqrt{(yy+x)}} + \frac{y\,dy}{\sqrt{(yy+x)}} = 0$, we have $A$ equal $\frac{x}{\sqrt{(yy-xx)}}$, $B$ equal $1 + \frac{y}{\sqrt{(yy+xx)}}$, $a = 1 + \frac{\frac{1}{2}}{\sqrt{(yy+x)}}$, and $b = \frac{y}{\sqrt{(yy+x)}}$

Let $x$ be ſuppoſed equal 5, and $y = 6$; then will $R$ equal $-.68$, $r = -.6$, $A$ equal $-1.5$, $B$ equal 2. 8, $a$ equal 1. 1, $b$ equal 9; therefore $\frac{Br - bR}{Ab - aB} = .23$, and $\frac{aR - Ar}{Ab - aB} = .37$, and the new Values of $x$ and $y$ equal to 5.23, and 6.37 reſpectively; the next Values will come out 5, 23263, and 6, 36898.

LXIV.

Let $49 \times (x - \frac{x}{(x+y)^2}) - 25 \times (1 - \frac{xx}{(1+y)^2}) = 0$, and $81 \times (1 - \frac{xx}{(1+y)^2}) - 49 \times (\frac{x}{y} - \frac{xy}{(1+x)^2}) = 0$. Here we have $A$ equal $49 \times (1 + \frac{x-y}{(x+y)^3} + \frac{50x}{(1+y)^2}$, $B = \frac{98x}{(x+y)^3} - \frac{50x^2}{(1+y)^3}$, $a = -\frac{162x}{(1+y)^2} + 49 \times [\frac{y}{1+x} - \frac{1}{y} - \frac{2xy}{(1+x)^3}]$, and $b = \frac{162x^2}{(1+y)^3} + 49x \times [\frac{1}{yy} + \frac{1}{(1+x)^2}]$. Application of the foregoing method to another Example.

Suppoſe $x = .8$, and $y = .6$, then will be found $R = .45$, $r = 2, 66$, $A = 68$, $B = 20, 7$, $a = -131$, $b = 146$, and the next Values of $x$ and $y$ equal to ,799, and ,582, with which repeating the Operation, the next following will come out ,79912 and ,58138.

LXV.

When an Equation contains two unknown Quantities $x$ and $y$, and it is propoſed to expreſs the Root $y$ in a Series of Terms including the Quantity $x$ with the other known Quantities; to find the Value of $y$ in ſuch a converging Series, it is neceſſary to conſider $x$ either 1°. as very ſmall, or 2° as very great, or 3°. as differing very little from ſome given Quantity; that ſo by ſuppoſing $x$ to be very ſmall the Series may converge wherein the Root $y$ is expreſſed by a Progreſſion of Terms, in which the Dimenſions of $x$ increaſe in the Numerators, 2° by ſuppoſing $x$ to be very great that Series may converge, in which the Dimenſions of $x$ continually increaſe in the Denominators of the Terms, or that 3° by ſuppoſing $x$ to differ but very little from ſome given Quantity, ſome Letter as $z$ being ſubſtituted for that Difference, it may come in the Place of $x$ conſidered as very ſmall. Fundamental principle of the method of Series.

LXVI.

Thus if the Equation to be reſolved was $x^2y + ay^2 - ax = 0$, if $x$ be conſidered as very little, the Value of $y$ may exhibited in a Series having its Terms compoſed of the Powers of $x$ divided by thoſe of $a$ with their reſpective Coeficients. Since when $x$ is very little in reſpect of $a$, the Terms $x$, $\frac{x^2}{a}$, $\frac{x^3}{a^2}$,

$\frac{x^4}{a^3}$, $\frac{x^5}{a^4}$, &c. decreaſe very quickly. If $x$ vaniſhes in reſpect of $a$, the ſecond Term will vaniſh in reſpect of the firſt, ſince $\frac{x^2}{a} : x = x : a$, and after the ſame Manner $\frac{x^3}{a^2}$ vaniſhes in reſpect of the Term immmediately preceding it.

But if $x$ be conſidered as very great, the Value of $y$ may be exhibited in a Series whoſe Terms are compoſed of the Powers of $a$ divided by thoſe of $x$ with their reſpective Coeficients, ſince when $x$ is very great in reſpect of $a$, then $a$ is very great in reſpect of $\frac{a^2}{x}$, and $\frac{a^2}{x}$ in reſpect of $\frac{a^3}{x^2}$, ſo that the Terms $a$, $\frac{a^2}{x}$, $\frac{a^3}{x^2}$, $\frac{a^4}{x^3}$, $\frac{a^5}{x^4}$, &c. in this Caſe decreaſe very ſwiftly; in either Caſe, the Series converge ſwiftly that conſiſt of ſuch Terms; and a few of the firſt Terms will give a near Value of the Root required.

Diviſion of Series into aſcending and deſcending ones.

Whence it appears that the Roots of Equations involving two unknown Quantities may be exhibited by two Sorts of Series, whereof one converges ſo much ſooner as the unknown Quantity it contains is leſs, and the other converges ſo much ſooner as the unknown Quantity is greater. In the Terms of the former, which are called aſcending Series, the Exponents of the unknown Quantity continually increaſe, but continually decreaſe in the Terms of the latter.

LXVII.

An Equation loſes ſome of its terms when one of its unknown quantities is ſuppoſed indefinitely great or indefinitely little.

To determine the firſt Term of the Series, exhibiting the Value of an unknown Quantity, in an Equation that involves two unknown Quantities, as of $y$ in the Equation $x^2y + ay^2 - a^2x = 0$, the Analyſis aſſume together ſuch Terms of the Equation as will be found to become vaſtly greater than the other Terms, that is, which give a Value of $y$ which ſubſtituted for it in all the Terms of the Equation, ſhall raiſe the Dimenſions of the other Terms all above or all below the Dimenſions of the aſſumed Terms, according as $x$ is ſuppoſed to be vaſtly little or vaſtly great in reſpect of $a$.

LXVIII.

Method of finding the greateſt terms of an equation by way of exclufion.

To determine the Terms in any propoſed Equation that the Suppoſition of $x$ or of $y$ being indefinitely great or indefinitely little, will render vaſtly greater than the other Terms: I obſerve firſt, that one Term of the Equation cannot be ſuppoſed vaſtly greater than all the reſt, for all the other Terms would vaniſh in reſpect of it, and this Term then alone would be equal to nothing, which is abſurd. The greateſt Terms therefore of an Equation are at leaſt two in Number, but there is no Reaſon

why any two Terms may not be ſuppoſed the greateſt of the Equation, unleſs the Conſequences ariſing from this Suppoſition deſtroy the Suppoſition itſelf.

Thus in the propoſed Equation $x^2 y + a y^2 - a^3 x = 0$, $x$ being conſidered as indefinitely great, if $x^2 y$ and $a y^2$ be ſuppoſed to be the greateſt Terms of the Equation, then $x^2 y + a y^2 = 0$, or $x^2 y = -a y^2$, and dividing by $-ay$, $y = -\frac{x^2}{a}$ which ſubſtituted for $x$, the Equation becomes $\frac{x^4}{a} + \frac{x^4}{a} - a^2 x = 0$, where the aſſumed Terms $x^2 y + a y^2$ are of more Dimenſions than the other $a^2 x$, and conſequently may be ſuppoſed to be the greateſt Terms of the Equation. Application of this method to an example.

If $x^2 y$ and $-a^2 x$ be ſuppoſed to be the greateſt Terms of the Equation, then $x^2 y - a^2 x = 0$, which by tranſpoſing and dividing by $x^2$ gives $y = \frac{a^2}{x}$, which ſubſtituted for $y$, then $a^2 x + \frac{a^5}{x^2} - a^2 x = 0$, where the aſſumed Terms $x^2 y$, $-a^2 x$ are of more Dimenſions than $a y^2$, and conſequently may be ſuppoſed to be the greateſt Terms of the Equation.

Finally, if $a y^2$ and $-a^2 x$ be ſuppoſed to be the greateſt Terms of the Equation, then $a y^2 - a^2 x = 0$, wherefore $y = a^{1:2} x^{1:2}$ which ſubſtituted for $y$, the Equation becomes $a^{1:2} x^{5:2} + a^2 x - a^2 x$ where the aſſumed Terms are of leſs Dimenſions than the other $x^2 y$, and conſequently cannot be ſuppoſed to be the greateſt Terms of the Equation.

$x$ being ſuppoſed indefinitely little, if $a y^2$ and $-a^2 x$ be ſuppoſed to be the greateſt Terms of the Equation, then $a y^2 - a^2 x = 0$, which gives $y = a^{1:2} x^{1:2}$ which ſubſtituted for $y$, the Equation becomes $a^{1:2} x^{5:2} + a^2 x - a^2 x = 0$, where the aſſumed Terms $a y^2 - a^2 x$ are of leſs Dimenſions than $x^2 y$, and conſequently may be ſuppoſed to be the greateſt Terms of the Equation.

If $x^2 y + a y^2$ be ſuppoſed to be the greateſt Terms of the Equation, then $x^2 y + a y^2 = 0$, which gives $y = -\frac{x^2}{a}$, which ſubſtituted for $y$, the Equation becomes $\frac{x^4}{a} + \frac{x^4}{a} - a^2 x = 0$, where the aſſumed Terms $x^2 y$, $a y^2$ are of more Dimenſions than $-a^2 x$, and conſequently cannot be ſuppoſed to be the greateſt Terms of the Equation.

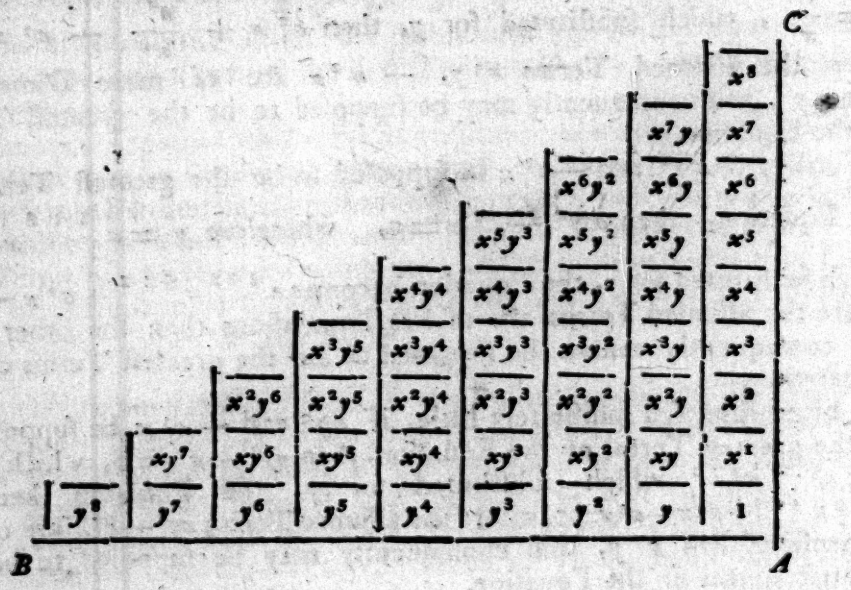
If $x^2 y$ and $-a^2 x$ be ſuppoſed to be the greateſt Terms of the Equation, then $y = \frac{a^2}{x}$ which ſubſtituted for $y$, the Equation becomes

$a^2x+\frac{a^3}{x^2}-a^2x=0$, where the assumed Terms $x^2y$ and $-a^2x$ are of more Dimensions than $ay^2$, consequently cannot be suppos'd to be the greatest Terms of the Equation; whence it appears that when $x$ is supposed indefinitely great, the proposed Equation $x^2y+ay^2-a^2x=0$, is reduced to the two following ones $x^2y+ay^2=0$, or $x^2+ay=0$, and $x^2y-a^2x=0$, or $xy-aa=0$; and when $x$ is supposed indifinitely little, it is reduced to $ay^2-a^2x=0$, or $y^2-ax=0$.

LXIX.

What was easy in an Equation consisting only of three Terms, would be very laborious if a more complex Equation was proposed, the Number of Terms occasioning a vast Number of Trials for the most Part unsuccessful. How the Analysts have remedied this Inconveniency, we shall therefore proceed to explain.

Use of the analytick triangle in this research.




Let $BA$ and $AC$ be drawn at Right Angles to each other, and let them be divided into equal Parts, and from the Points of Division let Lines be drawn meeting each other at Right Angles, which will form so many equal Squares as in the Figure; in these Squares place the Powers of $x$ from $A$ towards $C$, and the Powers of $y$ from $A$ towards $B$, and in any other Square place that Power of $y$ that is directly below it in the Line $AB$, so that the Index of $y$ in any Square may express its Distance from the Line $AC$, and the Index of $x$ in any Square may express its Distance from the Line $AB$.

Of this Figure or Triangle we are to obſerve, that being placed upon the Side $AB$, when $x$ is ſuppoſed indefinitely great, or indefinitely little, (and on the Side $AC$ when $y$ is ſuppoſed indefinitely great or indefinitely little) of all the Terms in the ſame vertical Column, that only can be conſidered as one of the greateſt Terms of the Equation, which is placed in the higheſt Square of this Column, when $x$ is ſuppoſed indefinitely great, or that which is placed in the loweſt Square of this Column, when $x$ is ſuppoſed indefinitely little; for $y$ having the ſame Exponent in all the Terms in the ſame Column, their Subordination entirely depends on the Exponent of $x$; wherefore $x$ being ſuppoſed indefinitely great, the greateſt Term in the Column is that in which $x$ has the greateſt Exponent, which is placed in the higheſt Square of this Column, and $x$ being ſuppoſed indefinitely little, the greateſt Term in the Column is that in which $x$ has the leaſt Exponent, or which is placed in the loweſt Square of this Column.

Property of the analytick triangle.

LXX.

The foregoing Obſervation, as is eaſy to perceive, ſerves to diminiſh the Number of Compariſons requiſite to be made, for diſcovering the greateſt Terms of an Equation, but in order to avoid every uſeleſs Compariſon, we are further to obſerve, that if every where for $y$, be ſubſtituted the Value expreſſed in the Powers of $x$, that ariſes for it, by ſuppoſing any two Terms of the Equation equal, the Dimenſions of $x$ in all the Terms that are found in the ſame ſtraight Line will be equal, but the Dimenſions of $x$ in the Terms above that Line will be greater than in thoſe in that Line, and the Dimenſions of $x$ in the Terms below the ſaid Line will be leſs than its Dimenſions in that Line.

The exponents of the terms which are in the ſame ſtraight line are in arithmetical progreſſion.

Thus if the two Terms $x^2$ and $x^3 y^2$ be ſuppoſed equal, we find $y^2 = \frac{1}{x}$, and $y = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$, and ſubſtituting this Value for $y$ in all the Squares, the Dimenſions of $x$ in the Terms, $x^4 y^4$, $x^5 y^6$, $x^6 y^8$, &c. which are found in the ſame ſtraight Line with $x^2$ and $x^3 y^2$, will be 2, but the Dimenſions of $x$ in all the Terms above that Line will be more than 2, and in all the Terms below that Line will be leſs than 2. For Example, ſubſtituting the foregoing Value of $y$ in the Term $x^4 y^3$, which is above the Line, it will be transformed into $x^{5:2}$, whoſe exponent exceeds 2, and if it be ſubſtituted in the Term $x^3 y^3$ below the Line, this Term will be transformed into $x^{3:2}$ whoſe Exponent is leſs than 2.

LXXI.

The foregoing Property of the Analytick Triangle is founded upon this Principle, that the Squares whoſe Centers are in a ſtraight Line contain Terms in which the Exponents of $x$ and $y$ are in Arithmetical Progreſſion: which is manifeſt with reſpect to the Terms in the vertical Column $AC$, or horoſontal Line $AB$, and their Parallels, in the Terms of the ſame Column the Exponents of $y$ are the ſame, and thoſe of $x$ form

the Arithmetical Progreſſion 1, 2, 3, 4, &c. in the Terms upon the ſame Line, the Exponents of $x$ are the ſame, and thoſe of $y$ form the Arithmetical Progreſſion 1, 2, 3, 4, &c. As to the Terms taken in any oblique ſtraight Line, let us ſuppoſe that this ſtraight Line parting from the Center of a Square, traverſes $k$ Lines and $l$ Columns before it paſſes through the Center of another Square, it is manifeſt, that ſince the Squares are ranged uniformly, it muſt traverſe $k$ Lines and $l$ Columns before it paſſes thro' the Center of a third Square, and as many more before it can attain the fourth, and ſo on. Wherefore, ſince the Exponent of $x$ is increaſed by Unity in aſcending a Line, and the Exponent of $y$ is increaſed by Unity in traverſing a Column, from right to left, if the Term placed in the firſt Square be $x^{m}\,y^{n}$, that placed in the ſecond will be $x^{m+k}\,y^{n+l}$, that in the third $x^{m+2k}\,y^{n+2l}$, and ſo on, in which the Exponents of $x$ form an Arithmetical Progreſſion, $m$, $m+k$, $m+2k$, &c. and thoſe of $y$ the Progreſſion $n$, $n+l$, $n+2l$, &c.

Demonſtration of the foregoing property of the analytic triangle

LXXII.

Converſely, Terms as $x^{m}\,y^{n}$, $x^{m+k}\,y^{n+l}$, $x^{m+2k}\,y^{n+2l}$, &c. in which the Exponents both of $x$ and of $y$ are in Arithmetical Progreſſion whoſe Differences are $k$ and $l$, thoſe Terms are placed in Squares whoſe Centers are under a Ruler traverſing at the ſame Time $k$ Lines and $l$ Columns, which determines the Inclination of this Ruler to the Sides of the Triangle.

The ratio of the differences of thoſe progreſſions depend on the inclination of the ruler to the ſides of the triangle.

For the Inclination of the Ruler to the Sides of the Triangle, and the Ratio of $k$ to $l$ depend entirely of each other, ſince $k$ and $l$ are the Number of Lines and the Number of Columns which the Ruler at the ſame Time traverſes. If $k$ ſurpaſſes $l$, the Ruler is more inclined to the Columns than to the Lines, and cuts off a greater Portion of the Side $AC$ of the Triangle, than of the Side $AB$. In general, ſince the Ruler traverſes $k$ Lines in traverſing $l$ Columns, it will traverſe, from Column to Column a Number of Lines expreſſed by $\frac{k}{l}$, whether $\frac{k}{l}$ denotes an Integer or a Fraction, if the Ruler traverſes two Columns in traverſing one Line, it will traverſe but Half a Line in traverſing one Column.

LXXIII.

The terms which are in two parallel ſtraight lines have exponents which form arithmetical progreſſions

Therefore if on the Surface of the Analitic Triangle there be drawn two parallel ſtraight Lines, and conſequently inclined as much one as the other to the Lines and Columns, the Squares whoſe Centers thoſe ſtraight Lines traverſe, contain Terms in which the Exponents of $x$ and $y$ are in Arithmetical Progreſſion, which in both one and the other ſtraight Line have the ſame Difference. Thoſe under the firſt ſtraight Line being, for Example, $x^{m}\,y^{n}$, $x^{m+k}\,y^{n+l}$, $x^{m+2k}\,y^{n+2l}$, &c. thoſe

under the second may be expressed by $x^{p}y^{q}, x^{p+k}y^{q+l}, x^{p+2k}y^{q+2l}$, &c. having the same common difference.

Conversely, if two Series of Terms be assumed as $x^{m}y^{n}, x^{m+k}y^{n+l}, x^{m+2k}y^{n+2l}$, &c. and $x^{p}y^{q}, x^{p+k}y^{q+l}, x^{p+2k}y^{q+2l}$, &c. in each of which the Exponents of $x$ and $y$ are in Arithmetical Progression, having the same common Difference, the straight Lines which pass through the Centers of the Cells of those two Series of Terms will be parallel.

LXXIV.

Such being the Disposition of the Terms in the analytick Triangle, if any two Terms be supposed equal, and a Ruler be applied to the Centers of the Cells of those two Terms. I say, that if in all the Terms placed in the Cells through whose Centers the Ruler passes, be substituted for $y$, the Value that arises for it from the foregoing Supposition, the Dimensions of $x$ in those Terms will be all equal. All the terms that are in the same straight line are of the same order, if two of them be supposed to be of the same order.

For as their Exponents are in Arithmetical Progression, they may be represented by the Series $x^{m}y^{n}, x^{m+k}y^{n+l}, x^{m+2k}y^{n+2l}, x^{m+3k}y^{n+3l}, x^{m+4k}y^{n+4l}, x^{m+5k}y^{n+5l}$, &c. Now let any two of them as $x^{m+2k}y^{n+2l}$, and $x^{m+5k}y^{n+5l}$ be supposed equal, then $\frac{x^{m+2k}y^{n+2l}}{x^{m+2k}y^{n+2l}} = \frac{x^{m+5k}y^{n+5l}}{x^{m+2k}y^{n+2l}}$, or $1 = x^{3k}y^{3l}$, consequently $x^{k}y^{l} = 1$, $x^{2k}y^{2l} = 1$, &c.

whence the Dimensions of $x$, in all the Terms under the Ruler $x^{m}y^{n}, x^{m+k}y^{n+l}, x^{m+2k}y^{n+2l}$, &c. resulting from the Multiplication of $x^{m}y^{n}$ into $1, x^{k}y^{l}, x^{2k}y^{2l}, x^{3k}y^{3l}$, &c. is $m - nk : l$, and the Dimensions of $x$ in the Terms $x^{p}y^{q}, x^{p+k}y^{q+l}$, &c. which are all found in the same straight Line parallel to the Ruler will be $p - qk : l$.

LXXV.

The Dimension of $x$ in all the Terms that are under the Ruler may be also found by examining where the Ruler cuts the first Column. If it passes through the Center of some Cell of this Column, it is manifest, that the Exponent of $x$ in the Term placed in this Cell will represent the Dimensions of $x$ in all the Terms under the Ruler, and if the Ru- Ratio of the orders of the two undetermined quantities in this supposition.

ler passes between the Centers of two Cells, the Point through which it passes will still serve to determine the Dimensions of $x$ in those Terms, whose Exponent in this Case is a Fraction. We are to conceive this Column, and in general each Column, as a straight Line divided into equal Parts by the Centers of the Cells; the Terms $x$, $x^2$, $x^3$, &c. whose Exponents are Integers, placed in the Points of Division, and the Terms, as $x^{1:2}$, $x^{1+1:3}$, $x^{1+3:4}$, &c. whose Exponents are Fractions or mixed Numbers, placed in the Points which divides the Intervals between the Centers in the same Ratio, as Unity is divided by the Fraction, which forms or concurs to form the Exponent of $x$. Hence we are to conceive $x^{1:2}$ placed precicely in the Middle betweeen $x^0$ or 1, and $x^1$, and $x^{1+1:3}$ placed in the first Point of the Subdivision of the Interval between $x^1$ and $x^2$ into three equal Parts, consequently if the Ruler passes, for Example, between the Centers of $x^2$ and of $x^3$, but three Times nearer the first than the last, that is, through the Point where $x^{2+3:4}$ is conceived to be placed, we are to conclude that the Dimensions of $x$ in all the Terms under the Ruler is $2\frac{3}{4}$.

Exponent of the order of the Terms that are in the same straight line.

The Exponent of the Dimensions of $x$ in the Terms under the Ruler is negative when it cuts the first Column produced below the first Line, which happens when $m < n\,k : l$, or $p < q\,k : l$, that is, when $\frac{k}{l} > \frac{m}{n}$ or $\frac{p}{q}$, then the Exponent $m - n\,k : l$ or $p - q\,k : l$ is negative. In this Case, the first Column is to be conceived as a straight Line produced below the Point, and divided into Parts equal to those that are above the Point, and the Terms whose Exponents are negative, as $x^{-1}$, $x^{-2}$, $x^{-3}$, &c. placed in the Points of Division.

The terms that are above this straight line are of a superior order: those that are below it of an inferior order.

LXXVI.

Since therefore of two parallel straight Lines drawn on the Surface of the analytick Triangle, that which is above the other cuts the first Column in a higher Point, the Terms in this Line are all of higher Dimensions of $x$ than those in the Line below it. Thus the Ruler passing through the Terms $x^{m}\,y^{n}$, $x^{m+k}\,y^{n+l}$, $x^{m+2k}\,y^{n+2l}$, &c. which are all of $m - n\,k : l$ Dimensions of $x$. Any other Term is of higher or lower Dimensions of $x$, according as it is placed in a Cell whose Center is above or below the Ruler.

LXXVII.

This is the Principle which determines the Comparisons to be made for finding the greatest Terms of an Equation. When $x$ is supposed

indefinitely great, it is manifeſt that it is uſeleſs to conſider two Terms as being the greateſt of the Equation, if the Ruler, applied to the Centers of their Cells, leaves above them any other Term, for this Term being of more Dimenſions of $x$ than thoſe through which the Ruler paſſes, it would therefore be indefinitely greater than thoſe which were ſuppoſed to be the greateſt, which is abſurd.

When $x$ is ſuppoſed indefinitely little, it is uſeleſs to ſuppoſe any two Terms to be the greateſt of the Equation, if the Ruler, applied to the Centers of their Cells leaves below them any Term of the Equation, for this Term being of lower Dimenſions of $x$ than thoſe through which the Ruler paſſes, it would be indefinitely greater than thoſe which were ſuppoſed to be the greateſt, which is abſurd.

LXXVIII.

Whence is deduced the following Rule, for diſcovering what Terms ought to be aſſumed from an Equation, in order to give a Value for $x$, or $y$, which ſhall make the other Terms, all of higher, or all of lower Dimenſions of $x$ or of $y$, than the aſſumed Terms.

Rule derived from the foregoing principles, for diſcovering the greateſt terms of an equation.

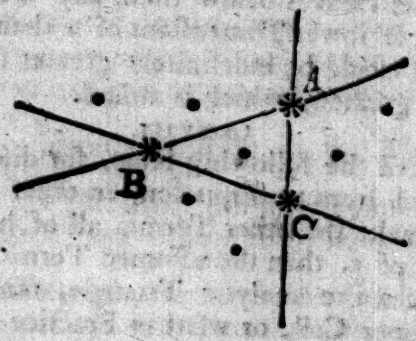
After having traced the analytic Triangle, range each Term of the Equation in its proper Cell, or what in Practice is more commodious, form a Triangle with Points diſpoſed in Quincunx, then change into an Aſteriſk each Point that repreſents a Square, which contains the ſame Dimenſions of $x$ and $y$, as the Terms in the Equation. Or a Triangle may be conſtructed of Wood or of Ivory, pierced with ſmall Holes and ranged at equal Diſtances, and parallel to the Sides of the Triangle, and the Holes which repreſent the Cells in which the Terms of the Equation are placed, may be ſtopped with Pegs.

Then the Triangle being couched on the Line without $x$, if $x$ be ſuppoſed indefinitely great or indefinitely ſmall, or on the Line without $y$, if $y$ be ſuppoſed indefinitely great or indefinitely ſmall, aſſume when the variable Quantity is ſuppoſed indefinitely great, ſuch Terms as lie in a ſtraight Line, ſo that the other Terms fall all below the ſtraight Line. This ſtraight Line which thus determines the greateſt Terms of the Equation, is called by the Analyſts, a *Superior Determinator*. The ſame Operation may furniſh ſeveral of them.

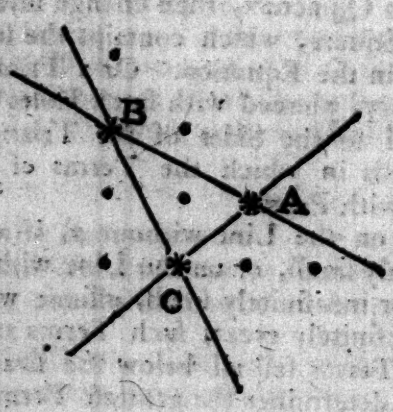
But if $x$ or $y$ be ſuppoſed indefinitely little, aſſume ſuch Terms as lie in a ſtraight Line, ſo that the other Terms lie all above the ſtraight Line, this ſtraight Line or thoſe ſtraight Lines, for there may be ſeveral of them, are called by the Analiſts, *Inferior Determinators*, becauſe they determine the greateſt Terms of the Equation, being thoſe which are placed in the Cells through whoſe Centers they paſs.

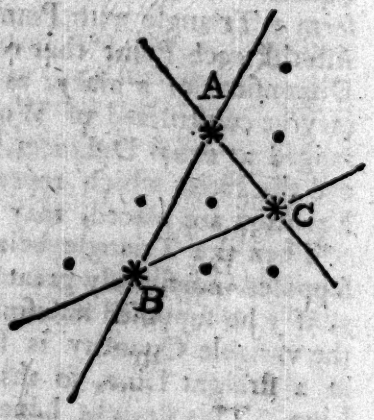
LXXIX.

Let the Equation propoſed be $x^2 y + a y^2 - a^2 x = 0$, having deſcribed the Triangle with Points, and converted into Aſteriſks the Points which repreſent the Cells $x^2 y$, $y^2$ and $x$, it will appear




Application of the foregoing rule to an example




that there are only three Determinators, $AB$, $BC$, $CA$, of which, when the Triangle is couched on the Line without $x$, two $AB$, $AC$, are ſuperior, and one inferior $BC$; but when the Triangle is couched on the Line without $y$, the Determinator $AB$ is a ſuperior one, and $AC$ and $BC$ are inferior ones.

The Determinator $AB$ gives the Equation $x^2 y + a y^2 = 0$, or $x x + a y = 0$, the Determinator $AC$ gives $x^2 y - a^2 x = 0$, or $x y - a a = 0$, and the Determinator $BC$ gives $a y^2 - a^2 x = 0$, or $y y - a x = 0$.

Whence $x$ being ſuppoſed indefinitely great, the propoſed Equation is reduced to thoſe two $xx + ay = 0$, and $xy - aa = 0$, given by the Determinators $AB$, $AC$, which are ſuperior ones, when the Triangle is couched on the Line without $x$.

But if $x$ is ſuppoſed indefinitely little, the Equation is reduced to $yy - ax = 0$, given by the Determinator $BC$, which is an inferior one, in the ſame Poſition of the Triangle.

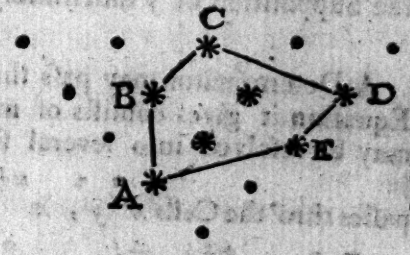
If $y$ be ſuppoſed indefinitely great, the Equation is reduced to $xx + ay = 0$, given by the Determinator $AB$, which is a ſuperior one, when the Triangle is couched on the Line without $y$.

But if $y$ be ſuppoſed indefinitely little, the Equation will be reduced to $xy - aa = 0$, and $yy - ax = 0$, given by the Determinators $AC$, $BC$, which are inferior ones in this ſame Poſition of the Triangle.

LXXX.

Let $xxyy + axy^2 + bx^2y + cx^3 + ddxy + eexx + f^3y = 0$, be propoſed.

After having formed the Triangle with Points, and converted into Aſteriſks the Points which correſpond to the Terms of the Equation, it will appear that all the Aſteriſks may be included in the Pentagon $ABCDE$, there are therefore five Determinators, which give the five following Equations.




Application of the foregoing rule to another example.

$AB$ gives $f^3y + axy^2 = 0$, or, dividing by $ay$, $xy + \frac{f^3}{a} = 0$.

$BC$ gives $axy^2 + x^2y^2 = 0$, or, dividing by $xy^2$, $x + a = 0$.

$CD$ gives $x^2y^2 + cx^3 = 0$, or, dividing by $x^2$, $yy + cx = 0$.

$DE$ gives $cx^3 + eex^2 = 0$, or, dividing by $cx^2$, $x + \frac{ee}{c} = 0$.

And $EA$ gives $eex^2 + f^3y = 0$, or, dividing by $ee$, $xx + \frac{f^3}{ee}y = 0$.

If $x$ be ſuppoſed indefinitely great, the Triangle being couched on the Line without $x$, then $CD$ is the only ſuperior Determinator. Wherefore this Suppoſition reduces the propoſed Equation to $yy + cx = 0$.

If $x$ be ſuppoſed indefinitely little, the Triangle remaining in the ſame Poſition, the inferior Determinators are $AB$ and $AE$, which give the

Equations $xy + \frac{f^2}{a} = 0$, and $xx + \frac{f^2}{a}y = 0$, to which two Equations the propoſed one is reduced by ſuppoſing $x$ indefinitely ſmall.

If $y$ be ſuppoſed indefinitely great, the Triangle being couched on the Line without $y$, there are three ſuperior Determinators $AB$, $BC$, $CD$, which give the Eqations $xy + \frac{f^2}{a} = 0$, $x + a = 0$, $yy + cx = 0$, to which the propoſed Equation is reduced by ſuppoſing $y$ indefinitely great.

But if $y$ be ſuppoſed indefinitely little, there are two inferior Determinators, $AE$, $ED$, which give the Equations, $xx + \frac{f^2}{ee}y = 0$, and $x + \frac{ee}{c} = 0$, to which the propoſed Equation is reduced by the Suppoſition of $y$ indefinitely little.

LXXXI.

A Determinator may paſs through more than two Cells, and then the Equation it gives conſiſts of more than two Terms, but this Equation may be reſolved into ſeveral ſimple Equations. If the Determinator paſſes thro' the Cells $x^{m}y^{n}$, $x^{m+k}y^{n+l}$, $x^{m+2k}y^{n+2l}$, &c. it will give

Roots of the equation given by a determinator.

$ax^{m}y^{n} + bx^{m+k}y^{n+l} + cx^{m+2k}y^{n+2l} + dx^{m+3k}y^{n+3l}$, &c. $= 0$, in which Equation, the Terms correſponding to the empty Cells will have their Coeficents $a$, $b$, $c$, $d$, &c. $= 0$. All the Terms of this Equation being diviſible by $x^{m}y^{n}$, it may be reduced to $a + bx^{k}y^{l} + cx^{2k}y^{2l} + dx^{3k}y^{3l}$, &c. $= 0$, or ſuppoſing $x^{k}y^{l} = z$, to $a + bz + czz + dz^3$, &c. $= 0$.

Let, $R$, $r$, $p$, &c. expreſs the Roots of this Equation, it may therebe reſolved into thoſe Equations $z - R = 0$, $z - r = 0$, $z - p = 0$, &c. That is, $x^{k}y^{l} - R = 0$, $x^{k}y^{l} - r = 0$, $x^{k}y^{l} - p = 0$, &c. which are reduced to $y = Rx^{-k:l}$, $y = rx^{-k:l}$, $y = px^{-k:l}$, &c. or infine to $y = Rx^{1:l-k:l}$, $y = rx^{1:l-k:l}$, $y = px^{1:l-k:l}$, &c.

LXXXII.

Thoſe Coeficients $R, r, p$, &c. of the Equations $y = R\,x^{1:l}$, $y = r\,x^{1:l}$, $y = p\,x^{1:l}$, &c. may be imaginary, they are all ſo, when the Roots of the Equation $a + bz + czz + dz^3$, &c. $= 0$, are imaginary, which may happen as often as the Equation is of an even Number of Dimenſions, when the compleat Number of its Terms is odd, when the Determinator paſſes through an odd Number of Cells, reckoning from the firſt full Cell to the laſt. But when this Number of Cells is even, the Equation $a + bz + czz$, &c. $= 0$, being of an odd Number of Dimenſions, has at leaſt one real Root. And in particular its Roots cannot be imaginary, when the Determinator traverſes only two full Cells, which are in two contiguous horizontal or vertical Columns. For in this Caſe, the Equation being $ax^m y^n + bx^{m+k} y^{n+l} = 0$, ($k$ or $l$ being equal to Unity, on account of the Contiguity of the Columns) we will have $a + bx^k y^l = 0$, or $a + bx^k y = 0$, that is, $x = -\frac{a}{b} y^{-l}$, or $y = -\frac{a}{b} x^{-k}$. They may be imaginary.

LXXXIII.

The Coeficents $R, r, p$, &c. may be real, and the Quantities $R\,x^{1:l}$, $r\,x^{1:l}$, $p\,x^{1:l}$, &c. nevertheleſs imaginary, or half imaginary. A Root is ſaid to be *half Imaginary*, ſuch as $\sqrt{ax}$, which is real when $x$ is poſitive, and imaginary when $x$ is negative, and is ſaid to be entirely imaginary, as $\sqrt{-xx}$, which is imaginary, whatever Value poſitive or negative is given to $x$. Or half imaginary.

Since the even Powers of a poſitive or negative Root, are neceſſarily poſitive, but the odd Powers are poſitive, if the Root is poſitive, and negative if the Root is negative, it is manifeſt that an odd Root is always real, whatever is the Power of the Quantity whoſe Root is extracted, but that an even Root can only be real when the Power is poſitive. Therefore if this Power is an even Power of a variable Quantity, the even Root will be real or imaginary, according as the Power is taken poſitively or negatively, that is, according as it is affected with a poſitive or negative Coeficent, but if the Power, whoſe even Root is extracted is an odd Power of a variable Quantity, the Root is half imaginary. Thus in the Equation $y = R\,x^{1:l} = \sqrt{Rx^l}$, if $l$ is an odd Num-

ber, $y$ is always a real Quantity, but if $l$ is an even Number, $y$ is half imaginary, $k$ being odd; and $k$ being even, $y$ is real when $R$ is positive, and imaginary when $R$ is negative.

LXXXIV.

We are further to obſerve with reſpect to the Exponent $-\frac{k}{l}$ of $x$, in the Equation $y = R^{1:l} x^{-k:l}$ given by the Determinator.

1°. That it is negative, when $k$ and $l$ have the ſame Sign, which happens when the Arithmetical Progreſſions $m$, $m+k$, $m+2k$, &c. $n$, $n+l$, $n+2l$. &c. of the Exponents of $x$ and of $y$ in the Terms which are under the Ruler, (Art. LXXI.) are both increaſing or both decreaſing. Then the Ruler recedes at the ſame Time from the firſt horizontal Line, and from the firſt vertical Line; it only cuts one of thoſe two Lines, or ſets out from the Point of the analytick Triangle. In this Caſe the Suppoſition of $x$ indefinitely great, renders $y = R^{1:l} x^{-k:l}$ or $\left(\frac{R}{x^k}\right)^{1:l}$ indefinitely little, and the Suppoſition of $x$ indefinitely little, renders $y$ indefinitely great.

Obſervations reſpecting the exponent of thoſe Roots.

2°. That this Exponent $-\frac{k}{l}$ is poſitive, when $l$ and $k$ have contrary Signs, which happens as often as one of the two Arithmetical Progreſſions $m$, $m+k$, $m+2k$, &c. $n$, $n+l$, $n+2l$, &c. is increaſing and the other decreaſing, when the Ruler approaches one of the Sides of the Triangle whilſt it recedes from the other, when it cuts them both any where elſe, but in the Point of the Triangle.

In this Caſe, the Suppoſition of $x$ indefinitely great or indefinitely little, will render alſo $y = R^{1:l} x^{k:l}$ or $\left(Rx^k\right)^{1:l}$ indefinitely great or indefinitely little. They are of the ſame Number of Dimenſions if $\frac{k}{l} = 1$, if $k = l$, if the Determinator is equally inclined to the two Lines. But if $\frac{k}{l} > 1$, if $k > l$, if the Determinator is more inclined to the vertical Columns than to the horizontal ones, retrenches a greater Portion of the Line without $y$, than of the Line without $x$, then $y = R^{1:l} x^{k:l}$ is of a higher Order than $x$, and of a lower Order than $x$; if $\frac{k}{l} < 1$, if $k < l$, if the Determinator retrenches a

leſs Portion of the Side of the Triangle without $y$, than of the Side without $x$.

3°. If $k = 0$, which happens when the Determinator is parallel to the Line without $x$, then $y = R^{1:l} x^{-k:l}$ is reduced to $y = R^{1:l} x^{0} = R^{1:l}$. whcrefore $x$ being ſuppoſed indefinitely great, gives only finit Values for $y$.

4°. And for a ſimilar Reaſon, when the Determinator is parallel to the Side of the Triangle without $y$, reduces $l$ to 0, it is obvious that $y$ being ſuppoſed indefinitely great, gives for $x$ only finit Values determined by the Roots of the Equation $a x^{m} y^{n} + b x^{m+k} y^{n+l} + c x^{m+2k} y^{n+2l}$, &c. $= 0$, which ſince $l = 0$, is reduced, by dividing by $x^{m} y^{n}$ to $a + b x^{k} + c x^{2k}$, &c. $= 0$.

LXXXV.

Thoſe Particulars reſpecting the Method of finding the greateſt terms of an Equation being premiſed, let the Value of $y$ deduced from any propoſed Equation involving $x$ and $y$ be expreſſed by the Series $Ax^{b} + Bx^{i} + cx^{k} + Dx^{l}$, &c. in which the Exponents $b$, $i$, $k$, $l$, &c. increaſe or decreaſe according as the Series is an aſcending or deſcending one. $A$, $B$, $C$, $D$, &c. are the Coeficents of the ſucceſſive Terms, and as it is poſſible that one or other of them may vaniſh with all thoſe that follow, it may happen that the Number of Terms of the Series is finite, and then it expreſſes the the exact Value of $y$.

LXXXVI.

To determine as many Terms of the Series as you pleaſe, proceed thus. To find the firſt Term. Suppoſe $x$ indefinitely great, if a deſcending Series is required, but indefinitely ſmall, if an aſcending Series is required. This Suppoſition reduces the Series to its firſt Term $Ax^{b}$, for if the Series is a deſcending one, the Exponent $b$ is greater than $i$, $k$ or $l$, &c. and $x$ being ſuppoſed indefinitely great, the Power $x^{b}$ is indefinitely greater than the others $x^{i}$, $x^{k}$, $x^{l}$, &c. (Art. LXXVI.) which may be neglected without Error, and if the Series is an aſcending one, the Exponent $b$ is the leaſt, and $x$ being indefinitely ſmall, the Power $x^{b}$ will

Inveſtigation of the ſucceſſive terms of a ſeries.

make all the others vaniſh, therefore the Suppoſition of $x$ indefinitely great, in a deſcending Series, and that of $x$ indefinitely little, in an aſcending Seires, reduces it to $y = Ax^{b}$. But thoſe ſame Suppoſitions reduce the propoſed Equation to one or ſeveral Equations, ſuch as $y = R^{1:l} x^{-k:l}$ given by the ſuper. or infer. Determinators, (Art. LXXVIII.) wherefore $A = R^{1:l}$ and $k = -\frac{k}{l}$, the Determinators therefore ſerve to determine the firſt Term of the Series, or Series, when the propoſed Equation furniſh ſeveral.

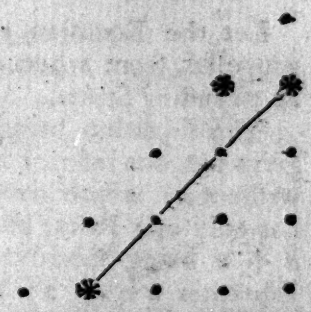
The following Terms are found after the ſame Manner. Let $u$ expreſs the Sum of the Terms $Bx^{i} + Cx^{k} + Dx^{l}$, &c. which ſucceed the firſt, conſequently $y = Ax^{b} + u$, this Value of $y$, ſubſtituted in the propoſed Equation, transforms it into another, including the variable Quantities $u$ and $x$; let $x$ be ſuppoſed in this Equation indefinitely great, when the Series required is a deſcending one, and indefinitely little, when the Series required is an aſcending one: and the ſuperior or inferior Determinators will give one or ſeveral Equations, ſuch as $u = R^{1:l} x^{-k:l}$ (Art. LXXIX.) but the ſame Suppoſitions of $x$ indefinitely great or indefinitely ſmall, reduces the Series $u = Bx^{i} + Cx^{k}$ &c. to $u = Bx^{i}$, therefore $B = R^{1:l}$ and $i = -\frac{k}{l}$, hence the Determinators of this firſt transformed Equation will give the ſecond Term $Bx^{i}$ of the Series.

Transforming a new the Equation, ſuppoſing $u = Bx^{i} + t$, where $t$ expreſſes all the Terms $Cx^{k} + Dx^{l}$, &c. which follow the ſecond Term of the Series; and the Determinators of this ſecond transformed Equation will give the third Term $Cx^{k}$ of the Series, and proceeding after the ſame Manner, the fourth Term will be obtained, and the following Terms to the laſt, if the Number of Terms is finite, or at leaſt as many as the End propoſed may require, if the Number of Terms of of the Series be infinite.

But we are to obſerve in the Courſe of thoſe Operations, that the Nature of aſcending Series require that the Exponents of $x$ ſhould continually increaſe, and in deſcending Series thoſe Exponents ſhould continually decreaſe. Wherefore, tho' at the firſt Operation performed on the propoſed Equation, all the ſuperior Determinators are to be taken into Conſideration, to obtain all the deſcending Series, or all the inferior Determinators to obtain all the aſcending Series: In the ſubſequent Operations, no Attention is to be paid to the ſuperior Determinators which will give the ſame or a greater Exponent than was obtained by the precedent Operation, nor to the inferior Determinators which will give the ſame or a leſs Exponent than what was obtained by the foregoing Operation. If there are no other Determinators, the Courſe of the Operations is finiſhed and the Series is ended.

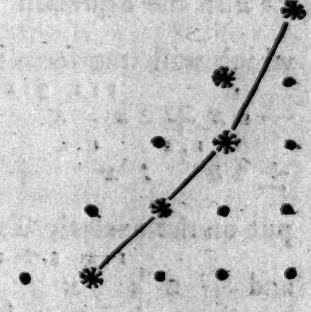
LXXXVII.

Let the Equation $a y^3 - x^3 y - a x^3 = 0$, be propoſed, and let it be placed on the analytic Triangle, and this Triangle being couched on the Line without $x$, there is but one inferior Determinator, which, paſſing thro' the Cells $y^3$ and $x^3$, gives the Equation $a y^3 - a x^3 = 0$, or $y = x$, which is the firſt Term of an aſcending Series. To find



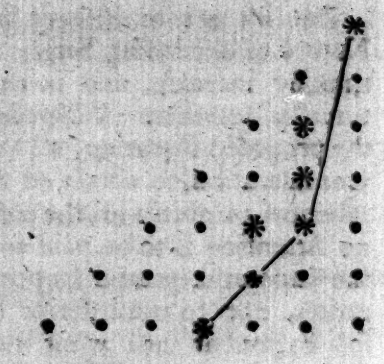
the ſecond, I ſubſtitute $x + u$ for $y$, and the Equation will be transformed into $a x^3 + 3 a u x x + 3 a u u x + a u^3 - x^4 - x^3 u - a x^3 = 0$, or $3 a u x x + 3 a u u x + a u^3 - x^4 - x^3 u = 0$, which being placed in its Turn on the analytic Triangle, gives two inferior Denominators, one which paſſes thro' the Cells $u^3$, $u u x$, $u x x$, is to be rejected, becauſe it would give $u = Rx$, this ſecond Exponent of $x$ being the ſame as was found by the firſt Operation; but the other Determinator, which paſſes thro' the Cells $u x x$ and $x^4$ gives the Equation $3 a u x x - x^4 = 0$, or $u = \frac{x x}{3 a}$, which is the ſecond Term of the Series. The third Term will be found by ſubſtituting $\frac{x x}{3 a} + t$ for $u$ in the foregoing Equation,

How to know when a ſeries becomes imaginary, half imaginary, or forked.



which transforms it into $x^4 + 3 a t x x + \frac{x^5}{3 a} + 2 t x^3 + 3 a t t x + \frac{x^6}{27 a a}$ $+ \frac{t x^4}{3 a} + t t x x + a t^3 - x^4 - \frac{x^3}{3 a} - t x^3 = 0$, or $3 a t x x + t x^3$ $+ 3 a t t x + \frac{x^6}{27 a a} + \frac{t x^4}{3 a} + t t x x + a t^3 = 0$,

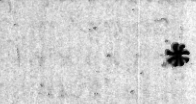
which being placed on the analytic Triangle, one Determinator gives $t = Rx$, and which consequently ſhould be rejected, but the other inferior Determinator which paſſes thro' $t\ x\ x$ and $x^6$, gives the Equation $3\,a\,t\,x\,x + \frac{x^6}{27\,a\,a} = 0$, or $t = -\frac{x^4}{81\,a^3}$, which is the third Term of the Series, for $y = x + u$, and $u = \frac{x\,x}{3\,a} + t$, and $t = -\frac{x^4}{81\,a^3}$, &c. hence $y = x + \frac{x\,x}{3\,a} - \frac{x^4}{81\,a^3}$ &c. the Law of the Progreſſion of the Terms being evident.

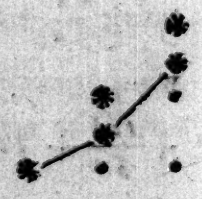


LXXXVIII.

Application of the foregoing method to an example.] Let the Equation $y\,y - 2\,x\,y + x\,x - 2\,a\,y + a\,x + a\,a = 0$ be propoſed, from whence is to be deduced the Value of $y$ in $x$ expreſſed by an aſcending Series.

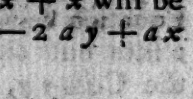
The Equation being placed on the analytic Triangle, gives but one inferior Determinator couched on the Line without $x$, which gives $y\,y - 2\,a\,y + a\,a = 0$, which has only one Root, but double, $y - a = 0$, or $y = a$; $a + u$ being ſubſtituted for $y$, gives the transformed Equation $a\,a + 2\,a\,u + u\,u - 2\,a\,x - 2\,u\,x + x\,x - 2\,a\,a - 2\,a\,u + a\,x + a\,a = 0$, or $u\,u - a\,x - 2\,u\,x + x\,x = 0$, placing it on the analytic Triangle, I find but one inferior Determinator, which gives the Equation $u\,u - a\,x = 0$, or $u = \pm\sqrt{a\,x} = \pm a^{\frac{1}{2}:2}\,x^{\frac{1}{2}:2}$. Subſtituting $\pm a^{\frac{1}{2}:2}\,x^{\frac{1}{2}:2} + t$ for $u$ in the Equation $u\,u - a\,x - 2\,u\,x + x^2 = 0$, which will transform it into $a\,x \pm 2\,a^{\frac{1}{2}:2}\,t\,x^{\frac{1}{2}:2} + t\,t - a\,x \mp 2\,a^{\frac{3}{2}:2}\,x^{\frac{1}{2}:2} - 2\,t\,x + x\,x = 0$, or $\pm 2\,a^{\frac{1}{2}:2}\,t\,x^{\frac{1}{2}:2} + t\,t \mp 2\,a^{\frac{1}{2}:2}\,x^{1+\frac{1}{2}:2} - 2\,t\,x + x\,x = 0$, which put on the analytic Triangle, and as two of its Terms $\pm 2\,a^{\frac{1}{2}:2}\,t\,x^{1+\frac{1}{2}:2}$ and $\mp 2\,a^{\frac{1}{2}:2}\,x^{1+\frac{1}{2}:2}$ have no Cells to lodge in, let them be placed (as explained Art. LXXV.) between two Cells, *viz.* the firſt in the ſecond Column between the Cells $t\,x^0$ or $t$, and $t\,x^1$, and the ſecond in the firſt Column between the Cell $x^1$ and the Cell $x^2$; there will then be two inferior Determinators, one paſſing thro' $t\,t$ and $t\,x^{\frac{1}{2}:2}$, giving $t = Rx^{\frac{1}{2}:2}$ is to be rejected, as having the ſame





Exponent as the precedent Term; the other Determinator paſſing thro' the Cells $t^{1:2}x$ and $x^{1+1:2}$, gives $\mp 2a^{1:2}t^{1:2}x + 2a^{1:2}x^{1+1:2} = 0$, or $t = x$, which is the third Term of the Series.

To obtain the fourth, $x + s$ is to be ſubſtituted for $t$ in the laſt Equation $\pm 2a^{1:2}t^{1:2}x + tt \mp 2a^{1:2}x^{1+1:2} - 2tx + xx = 0$, which will transform it into $\pm 2a^{1:2}x^{1+1:2} \pm 2a^{1:2}s^{1:2}x + x^2 + 2sx + ss \mp 2a^{1:2}x^{1+1:2} - 2xx - 2sx + xx = 0$, or into $\pm 2a^{1:2}s^{1:2}x + ss = 0$; thoſe Terms being placed on the analytic Triangle gives only one Determinator, which gives $s = Rx^{1:2}$, this Exponent being leſs than the preceding one, is to be rejected. Hence the Series is ended for $y = a + u = a \pm \sqrt{ax} + t = a \pm \sqrt{ax} + x$.



On examining what this laſt Determinator gives, or its Equation $\pm 2a^{1:2}s^{1:2}x + ss = 0$, we will find it has two Roots, 1°. $s = 0$, which terminates the Series; 2°. $s = \mp 2a^{1:2}x^{1:2} = \mp 2\sqrt{ax}$, which ſubſtituted in $y = a \pm \sqrt{ax} + x + s$, gives $y = a \pm \sqrt{ax} + x \pm 2\sqrt{ax}$, which is the ſame as $y = a \pm \sqrt{ax} + x$, which is therefore the exact Value of $y$, which may be eaſily verified, ſince by taking away the Irrationality, the Equation $y = a \pm \sqrt{ax} + x$ will be transformed into the propoſed Equation $yy - 2xy + xx - 2ay + ax + aa = 0$.

## LXXXIX.

Application of the foregoing method to another example.

It is to be obſerved, 1°. that if among the Equations which the ſucceſſive Determinators furniſh, there be found any one whoſe Roots are all imaginary, the whole Series which the firſt Terms ſeemed to announce will become imaginary, for one imaginary Term renders the whole Sum, of which it is a Part, imaginary, unleſs what is imaginary in one Term be deſtroyed by what is imaginary in another Term, which cannot happen in this Caſe, where $x$ in each Term has a different Exponent.

2°. That if among the Terms of a Series there be any which are half imaginary, the Series is half imaginary, that is, imaginary when $x$ is ſuppoſed negative, and real when $x$ is ſuppoſed poſitive or reciprocally.

3°. That if among the Equations which ſerve to determine the ſucceſſive Terms of a Series, there be found any which have ſeveral real Roots;

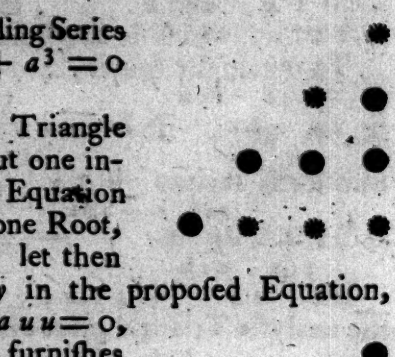
then the Series becomes forked as it were, and is multiplied into as many Series as there are real Roots, as often as that happens.

XC.

Let the Value of $y$ in $x$, expreſſed by an aſcending Series deduced from $x^3 + x^2 y + a yy - 2a^2 y + a^3 = 0$ be required.

Example of an imaginary ſeries.

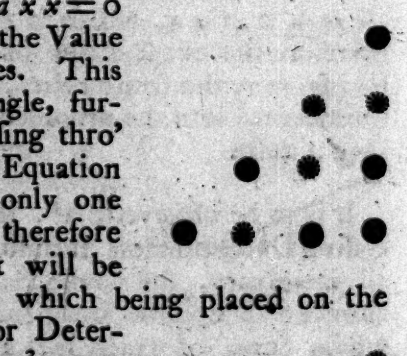
This Equation placed on the analytic Triangle couched on the Line without $x$, furniſhes but one inferior Determinator, which gives the Equation $ayy - 2a^2 y + a^3 = 0$, which has only one Root, but double, *viz.* $y - a = 0$, or $y = a$; let then $y = a + u$, and ſubſtituting this Value of $y$ in the propoſed Equation, it will transform it into $x^3 + ax^2 + ux^2 + auu = 0$, which being placed on the analytic Triangle, furniſhes but one inferior Determinator, which gives $auu + axx = 0$, whoſe Roots $u = \pm\sqrt{-xx}$ are imaginary; conſequently the Value of $y$ in $x$ cannot be expreſſed by an aſcending Series, becauſe the only one which could expreſs this Value would be $y = a + u = a \pm \sqrt{-xx}$, &c. which is imaginary.



XCI.

Example of a ſeries half imaginary.

Let the Equation $x^2 y + ayy - 2axy + axx = 0$ be propoſed, from whence is to be deduced the Value of $y$ in $x$, expreſſed by an aſcending Series. This Equation being placed on the analytic Triangle, furniſhes but one inferior Determinator, paſſing thro' the Cells $yy$, $xy$, $xx$, which gives the Equation $ayy - 2axy + axx = 0$, which has only one Root, but double, $y = x$. Subſtituting therefore $x + u$ for $y$ in the propoſed Equation, it will be transformed into $x^3 + x^2 u + auu = 0$, which being placed on the analytic Triangle, furniſhes but one inferior Determinator, which gives the Equation $auu + x^3 = 0$,

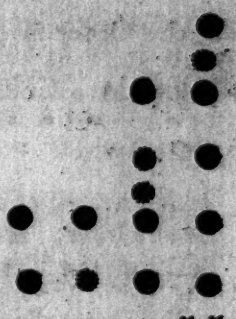


which has two Roots $u = +\sqrt{-\frac{x3}{a}}$ and $-\sqrt{-\frac{x3}{a}}$

ſubſtituting $\pm\sqrt{-\frac{x^3}{a}} + t = a^{-1:2} \times -x^{3:2} + t$

for $u$ in the firſt transformed Equation, the ſecond will be $\pm a^{-1:2} \times -x^{3+1:2} + txx \pm 2a^{1:2} t \times -x^{3:2} + att = 0$,

which being placed on the analytic Triangle, furniſhes two inferior Determinators, one uſeleſs, becauſe paſſing thro' the Cells $t\ t$, $t\,x^{3:2}$, it gives $t = R\,x^{3:2}$, where $x$ has the ſame Exponent as in the precedent Term; the other paſſing thro' $t\,x^{3:2}$ and $x^{3+1:2}$, gives the Equation $\pm 2a^{1:2} t \times - x^{3:2} \pm a^{-1:2} \times - x^{3+1:2} = 0$, or $t = \frac{xx}{2a}$



wherefore the three firſt Terms of the Series are $x \pm \sqrt{-\frac{x^3}{a}} - \frac{x^2}{2a}$.

From the Inſpection of which it appears, 1°. that the Series is imaginary, when $x$ is ſuppoſed poſitive, becauſe then $\sqrt{-\frac{x^3}{a}}$ is an imaginary Quantity; but if $x$ be ſuppoſed negative, the Series is real, and then

2°. The Series is double, becauſe the Term $\sqrt{-\frac{x^3}{a}}$ has equally the Sign + and the Sign —, the Equation $a\,u\,u + x^3 = 0$, which furniſhed this Term, having two real Roots $u = +\sqrt{-\frac{x^3}{a}}$, and $u = -\sqrt{-\frac{x^3}{a}}$ wherefore in reality there are two Series, the three firſt Terms of one are $x + \sqrt{-\frac{x^3}{a}} - \frac{xx}{2a}$ and of the other $x - \sqrt{-\frac{x^3}{a}} - \frac{xx}{2a}$.

XCII.

The general Method being attended with tedious Calculations, we ſhall now proceed to explain how the Analyſts have abridged them.

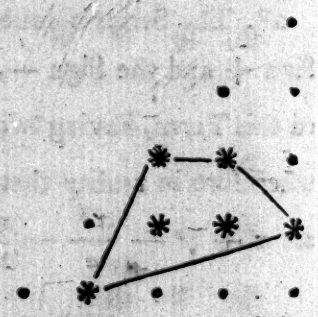
Manner the terms of a transformed equation are placed in the analytic triangle.

We are to obſerve, that the Subſtitution of $Ax^b + z$ for $y$ (Art. LXXXVI.) in any Term of the propoſed Equation, transforms it into as many Terms as there are Columns, from the firſt to that in which it is placed incluſively; each Term being placed in a different Column, and all thoſe Terms being ſituated on the ſame ſtraight Line, parallel to the Determinator which gave the Equation $y = Ax^b$.

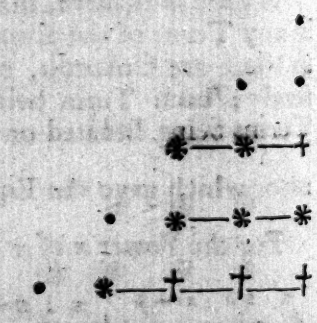
For the Power $n$ of $u + Ax^b$ being $u^n + nAx^b u^{n-1} + \frac{n \cdot n-1}{1 \cdot 2} A^2 x^{2b} u^{n-2} + \frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3} A^3 x^{3b} u^{n-3} +$, &c. to $A^n x^{nb}$, or the laſt Term, which being ſubſtituted for $y^n$ in a Term as $x^m y^n$

of the Order $m + n h$, $y$ being of the Order $h$ (Art. LXXIV.) and which is placed in a Column preceded by $n$ others, will transform it into $x^m u^n + n A x^{m+b} u^{n-1} + \frac{n.n-1}{1.2} A^2 x^{m+2b} u^{n-2} +$, &c. to $A^n x^{m+nb}$ which are all of the Order $m + n h$. But all the Terms which are of the same Order, are placed in a right Line, parallel to the Determinator hence all the Terms into which $x^m y^n$ is transformed are in a right Line parallel to the Determinator which gave $y = A x^b$; and it is evident, that the Term $x^m u^n$ ocupies the Cell in which the transformed Term $x^m y^n$ was lodged in a Column preceded by $n$ others; that the second Term $x^{m+b} u^{n-1}$ is in a Column preceded by $n - 1$ Columns; that the third $x^{m+2b} u^{n-2}$ is in the next Column, and so on to the last $A^n x^{m+nb}$ which is in the first Column, or in the Line of the Powers of $x$.

Example. Thus when the Equation $x^2 y^2 + a y^3 + b x y^2 + c x^2 y + d d x y + f^3 x = 0$ is placed on the analytic Triangle, couched on the Line without $x$, it will give four Determinators; first an horizontal one, which passes thro' the Cells $x^2 y^2$ and $x^2 y$, which gives for $y$ a constant Value (Art. LXXXIV.) & which may be expressed by $A$; substituting $A + u$ for $y$ in the proposed Equation, it will be transformed into $A^2 x^2 + 2 A u x^2 + u^2 x^2 + A^3 a + 3 A^2 a u + 3 A a u^2 + a u^3 + A^2 b x + 2 A b u x + b u^2 x + A c x^2 + c u x^2 + A d d x + d d u x + f^3 x = 0$, from whence it appears, that the Terms



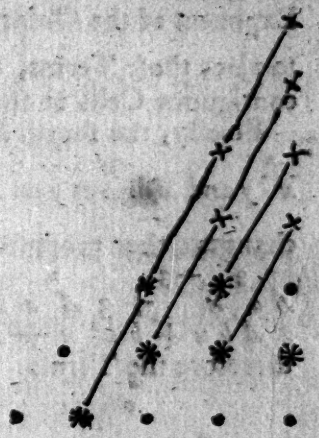
| | | |
|---|---|---|
| $x^2 y^2$ | of the proposed Equation, produce in the transformed Equation | $x^2 u^2, x^2 u, x^2$ |
| $y^3$ | | $u^3, u^2, u^1, u^0$ |
| $x y^2$ | | $x u^2, x u, x$ |
| $x^2 y$ | | $x^2 u, x^2$ |
| $x y$ | | $x u, x$ |
| $x$ | | $x$ |



Thus each Term has given one to each Column which precedes it, and those Terms are on an horizontal Line, that is, parallel to the Determinator which gave $y = A$.

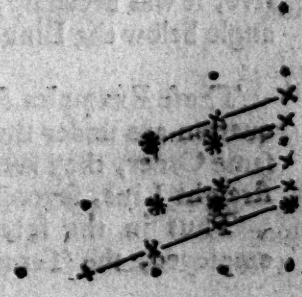
The ſecond Determinator of the propoſed Equation paſſed thro' the Cells $y^3$ and $x^2y^2$, and gave $y = Ax^2$: the Subſtitution of $Ax^2 + u$ for $y$, transforms the Equation into $A^2x^6 + 2Aux^4 + u^2x^2 + A^3ax^6 + 3Aaux^4 + 3Aau^2x^2 + au^3 + A^2bx^5 + 2Abux^3 + bu^2x + Acx^4 + cux^2 + Addx^3 + ddux + f^3x = 0$, therefore the Terms

$x^2y^2$, $y^3$, $xy^2$, $x^2y$, $xy$, $x$ } have given {
$x^2u^2$, $x^4u$, $x^6$
$u^3$, $x^2u^2$, $x^4u$, $x^6$
$xu^2$, $x^3u$, $x^5$
$x^2u$, $x^4$
$xu$, $x^4$
$x$



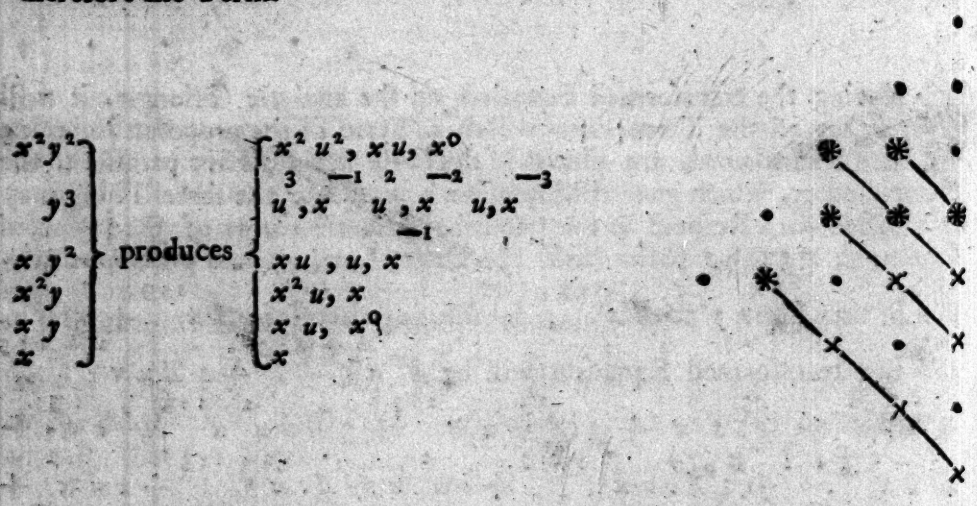
and placing the transformed Equation on the analytic Triangle, it will appear that all the Terms into which a Term of the propoſed Equation had been transformed, are placed in the ſame ſtraight Line parallel to the Determinator, which gave the Equation $y = Ax^2$, the ſame Thing may be verified with Reſpect to the two other Determinators of the propoſed Equation; the third paſſes thro' the Cells $y^3$ and $x$, and gives an Equation of this Form $y = Ax^{1:3}$; ſubſtituting therefore $Ax^{1:3} + u$ for $y$, and the transformed Equation will be $A^2x^{2+2:3} + 2Aux^{2+1:3} + u^2x^2 + A^3ax^3 + 3A^2aux^{2:3} + 3Aau^2x^{1:3} + au^3 + A^2bx^{2+2:3} + 2Abux^{1+1:3} + bu^2x^2 + Acx^{2+1:3} + cux^2 + Addx^{1+1:3} + ddux + f^3x = 0$, hence the Terms

$x^2y^2$, $y^3$, $xy^2$, $x^2y$, $xy$, $x$ } give {
$x^2ux^{2+1:3}$, $ux^{2+2:3}$
$u^3x^{1:3}$, $ux^{2:3}$, $ux^{1:3}$
$xux^{2+1:3}$, $ux^{1+2:3}$
$xu^2x^{1:3}$, $ux^{1+1:3}$
$xu$, $x$
$x$



This transformed Equation being placed on the analytic Triangle, it will appear that each Term of the propoſed Equation has given a Term to each Columa that precedes it; and that thoſe Terms are on a ſtraight Line, parallel to the Determinator which gave $y = A x^{1:2}$, but as the Exponent of the Order of $y$ is a Fraction $\frac{1}{3}$, it was neceſſary, in order to place thoſe Terms, to divide into three equal Parts the Intervals of the contiguous Cells in the ſame Column (Art. LXXV.)

Finally, the fourth Determinator of the propoſed Equation paſſing thro' $x^2$ $y$ and $x$, gives $y = A x^{-1}$, and the Subſtitution of $A x^{-1} + u$ for $y$, changes the propoſed Equation into $A^2 + 2 A u x + u u x x + A a x^3 + 3 A a u x^{-3} + 3 A a u u x^{-2} + a u^3 x^{-1} + A b x^2 + 2 A b u x^{-1} + b u^2 x + A c x + c u x^2 + A d d + d d u x + f^3 x = 0$, therefore the Terms




Here the ſame Rule is obſerved, but as the negative Exponent (— 1) of the Order of $y$, introduces Terms in which the Exponent of $x$ is negative, it was neceſſary, in order to place thoſe Terms, to produce the Triangle below the Line without $x$ (Art. LXXV.)

XCIII.

Thoſe Examples ſhew, that when ſeveral Terms of the propoſed Equation are under the Determinator, or any of its Paralells, or are of the ſame Order, they are transformed into Terms, which ranged in the ſame ſtraight Line, ſeveral of them ſometimes lodge together in the ſame Cell.

Thus in the foregoing Example, if the horizontal Determinator be employed, the Terms $x^2 y^2$ and $c x^2 y$, which were under this Deter-

minator, were transformed into $x^2 u^2 + (2A + c) x^2 u + (A^2 + Ac) x^2$, and the Terms $b x y^2$, $d d x y$, $f^3 x$, which are under a ſtraight Line, parallel to this Determinator, have given $b x u^2 + (2 A b + d d) x u + (A^2 b + A d d + f^3) x$, which are ſtill under the ſame ſtraight Line.

Wherefore, when there are ſeveral Terms of the ſame Order, as $a x^m y^n + b x^{m+b} y^{n-1} + c x^{m+2b} y^{n-2}$, &c. continued to $x^{m+nb}$, which are all of the Order $m + n h$, ſince $y\ (= A x^b)$ is of the Order $h$, the Subſtitution of $A x^b + u$ for $y$ transforms it into a Series of Terms, ſuch as $P x^m u^n + Q x^{m+b} u^{n-1} + R x^{m+2b} u^{n-2} +$, &c. continued to $Z x^{m+nb}$, which alſo are all of the Order $m + n h$, in which the Exponents of $x$ form an arithmetical Progreſſion, the common Difference being $h$, and the Exponents of $u$ another arithmetical Progreſſion, the common Difference being 1.

The Coeficients $P$, $Q$, $R$, &c. of thoſe Terms, may be calculated by a contracted Rule, derived from the following Conſideration: The Term $a x^m y^n$, by the ſubſtitution of $u + A x^b$ for $y$, is transformed into $a x^m u^n + n a A x^{m+b} u^{n-1} + \frac{n \cdot n-1}{1 \cdot 2} a A^2 x^{m+2b} u^{n-2} + \frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3} a A^3 x^{m+3b} u^{n-3}$, &c. the Term $b x^{m+b} y^{n-1}$ is transformed into $b x^{m+b} u^{n-1} + \frac{n-1}{1} b A x^{m+2b} u^{n-2} + \frac{n-1 \cdot n-2}{1 \cdot 2} b A^2 x^{m+3b} u^{n-3}$, &c. the Term $c x^{m+2b} y^{n-2}$ into $c x^{m+2b} u^{n-2} + \frac{n-2}{1} c A x^{m+3b} u^{n-3}$, &c. and the Term $d x^{m+3b} y^{n-3}$ into $d x^{m+3b} u^{n-3}$, &c. Wherefore, the Sum $a x^m y^n + b x^{m+b} y^{n-1} + c x^{m+2b} y^{n-2} + d x^{m+3b} y^{n-3}$, &c. is transformed into $a x^m u^n + (n a A + b) x^{m+b} u^{n-1} + \left(\frac{n.n-1}{1 \cdot 2} a A^2 + \frac{n-1}{1} b A + c\right) x^{m+2b} u^{n-2} + \left(\frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3} a A^3 + \frac{n-1 \cdot n-2}{1 \cdot 2} b A^2 + \frac{n-2}{1} c A + d\right) x^{m+3b} u^{n-3}$, &c. from whence is deduced the following Rule:

Abridged method of performing the transformations requisite to be made for finding the terms of a series.

Write in the first Line all the Terms of the same Order, or even the whole Equation, distinguishing, for Conveniency's sake, the Orders of its Terms, changing, if you please, $y$ into $u$; I say if you please, for it will be found upon Trial, that it will be more commodious not to do it; but then it must be observed, that $y$, which denotes before the Operation the whole Series, and during the Operation its first Term only, denotes, during the second Operation, the second Term, and during the third Operation, the third Term, &c. Having wrote the proposed Equation in the first Line, and ranged the Terms according to their different Orders, multiply each Term by the Exponent of $y$ and by $\frac{b}{A}x$, and divide those Products by $y$, and this will give a second Line; multiply each Term of this second Line by half the Exponent of $y$ and by $\frac{b}{A}x$, and divide the Products by $y$, and a third Line will be obtained. Each Term of this Line, multiplied by one third of the Exponent of $y$ and by $\frac{b}{A}x$, and divided by $y$, will give a fourth Line. And this Operation is to be continued until there remains but Terms without $y$; then the Sum of all those Lines will be the transformed Equation.

XCIV.

Application of this method to an example.

Thus in the Equation of the foregoing Article, if we employ the horizontal Determinator, which gave the Equation $x^2y^2 + cx^2y = 0$, or $y = -c$, there will result $A = -c$ and $h = 0$; and the Operation is performed thus:

| | I. Order. | | II. Order. | | | III. Order. |
|---|---|---|---|---|---|---|
| | $x^2y^2 + cx^2y$ | | $+ bxy^2 + ddxy + f^3x$ | | | $+ ay^3$ |
| | 2 1 | | 2 1 0 | | | 3 |
| $x - c)$ | | | | | | |
| | $-2cx^2y - ccx^2$ | | $- 2bcxy - cddx$ | | | $- 3acyy$ |
| | $\frac{1}{2}$ 0 | | $\frac{1}{2}$ 0 | | | $\frac{2}{2}$ |
| | $+ ccxx$ | | $+ bccx$ | | | $- 3accy$ |
| | 0 | | 0 | | | $\frac{2}{3}$ |
| | | | | | | $- ac^3$ |

The transformed Equation therefore is $x^2y^2 + (c - 2c)x^2y + (cc - cc)x^2 + bxy^2 + (dd - 2bc)xy + (f^3 - cdd + bcc)x + ay^3 - 3acyy + 3accy - ac^3 = 0$, in which the second Term is reduced to $-cx^2y$, and the third vanishes.

If in the ſame Equation we would employ the Determinator that paſſes thro' the Cells $c x^2 y$ and $f^3 x$, which gives $y = -\frac{f^3}{cx}$ the Operation will be performed thus: Application of this method to another example.

I.Order. II.Order. III.Order. IV.Order.

$$\overbrace{c x^2 y + f^3 x}^{1\ \ 0} + \overbrace{x^2 y^2 + d d x y}^{2\ \ 1} + \overbrace{b x y y}^{2} + \overbrace{a y^3}^{3}$$

$$\times -\frac{f^3}{cx}$$

$$-f^3 x \quad -\frac{2 f^3 x y}{c} \quad -\frac{f^3 d d}{c} \quad -\frac{2 b f^3 y}{c} \quad -\frac{3 a f^3 x y}{c x}$$
$$0 \qquad \frac{1}{2} \qquad 0 \qquad \frac{1}{2} \qquad \frac{2}{3}$$

$$+\frac{f^6}{cc} \quad +\frac{b f^6}{cc x} \quad -\frac{3 a f^6 y}{cc x x}$$
$$0 \qquad 0 \qquad \frac{1}{3}$$

$$+\frac{a f^9}{c^3 x^3}$$

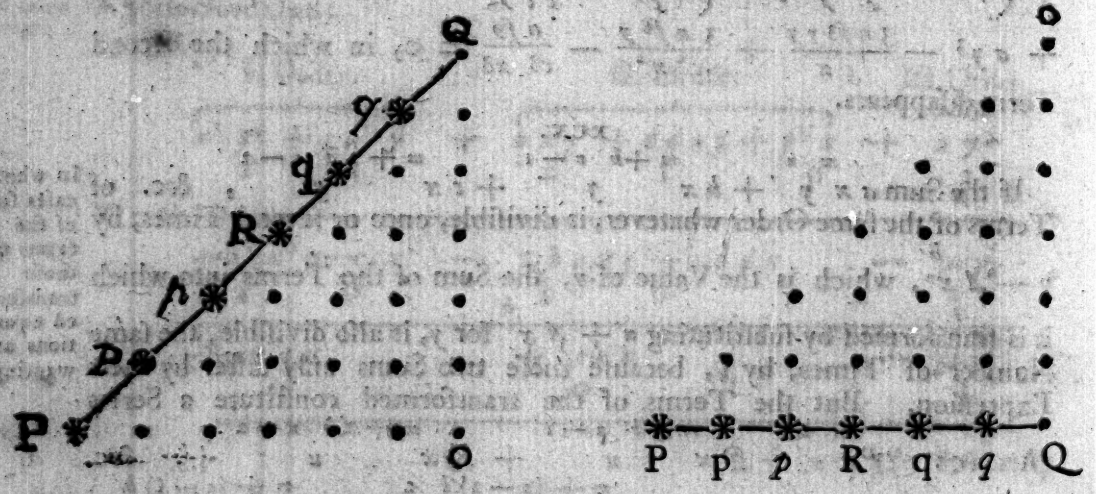
Whence the transformed Equation is $c x^2 y + (f^3 - f^3) x + x^2 y^2 + \left(d d - \frac{2 f^3}{c}\right) x y - \left(\frac{f^3 d d}{c} - \frac{f^6}{cc}\right) + b x y y - \frac{2 b f^3 y}{c} + \frac{b f^6}{cc x} + a y^3 - \frac{3 a f^3 y y}{c x} + \frac{3 a f^6 y}{c^4 x^2} - \frac{a f^9}{c^3 x^3} = 0$, in which the ſecond Term diſappears.

## XCV.

If the Sum $a x^m y^n + b x^{m+b} y^{n-1} + c x^{m+2b} y^{n-2}$, &c. of Terms of the ſame Order whatever, is diviſible, once or ſeveral Times, by $y - A x^{\frac{b}{}}$, which is the Value of $u$, the Sum of the Terms into which it is transformed by ſubſtituting $u + A x^{\frac{b}{}}$ for $y$, is alſo diviſible, the ſame Number of Times, by $u$, becauſe thoſe two Sums only differ by their Expreſſion. But the Terms of the transformed conſtitute a Series (Art. XCII.) $P x^m u^n + Q x^{m+b} u^{n-1} + R x^{m+2b} u^{n-2}$, &c. terminated by the Terms $+ X x^{m+(n-2)b} u^2 + Y x^{m+(n-1)b} u + Z x^{m+nb}$. This Series cannot be diviſible by $u$, unleſs the laſt Term $= 0$ and $Z = 0$; it cannot be diviſible by $u u$, or twice by $u$, unleſs the two laſt Terms $Y$ and $Z$ be $= 0$; it cannot be diviſible by $u^3$, or three Times by $u$, unleſs $X$, $Y$, and $Z$ be $= 0$. In general, as many Times: In what caſes ſome of the terms of thoſe transformed equations are wanting.

as $u$, or rather $y - Ax^b$, which is its Value, divides the Sum of the Terms of any Order, ſo many Terms of this Order are wanting in the transformed in the firſt Columns, for the Terms $Z$, $Y$, $X$, &c. are thoſe which are placed in the firſt, ſecond, third, &c. Columns.

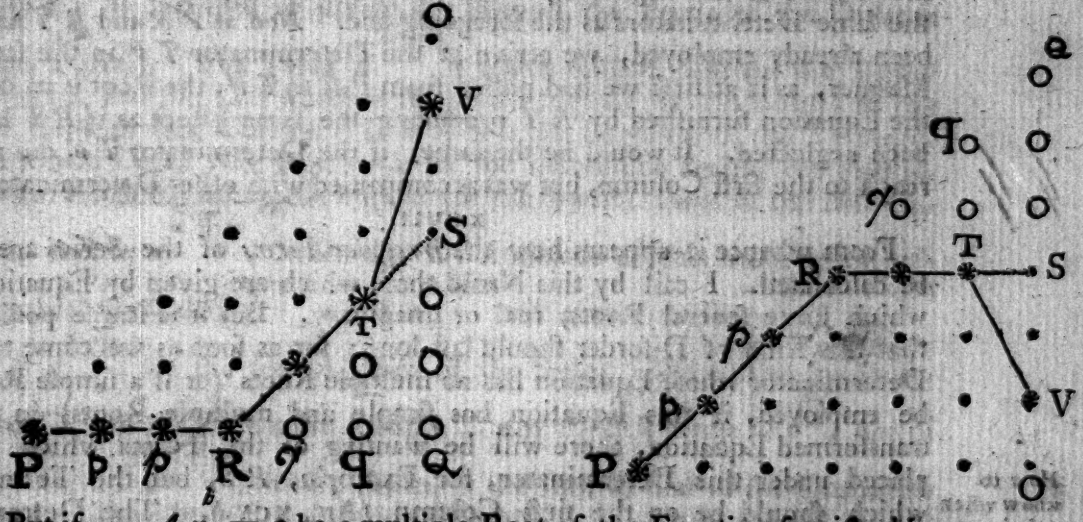
Wherefore, ſince $y - Ax^b = 0$ is one of the Roots furniſhed by the Determinator, $y - Ax^b$ divides at leaſt once, the Sum of the Terms which are under this Determinator; conſequently, in the transformed Equation is neceſſarily wanting the Term correſponding to the Point where the Determinator cuts the firſt vertical Line, and in the transformed Equation, will be wanting the two Terms correſponding to the Points where the Determinator cuts the firſt and ſecond Column, if $y - Ax^b$, divides the Sum of the Terms which are under the Determinator, twice, if $y - Ax^b = 0$ be a double Root of the Equation which this Determinator furniſhes; but if $y - Ax^b = 0$ be a triple Root of this Equation, in the transformed Equation will be wanting the Terms correſponding to the Points where the Determinator croſſes the three firſt Columns, and ſo on.




XCVI.

If therefore $PQ$ repreſents a Determinator, and $y - Ax^b = 0$ be a ſimple Root of the Equation which it furniſhes, in the transformed Equa-

tion under this Determinator will be wanting only the Term which ſhould occupy the Cell Q in the firſt Column Q O, at leaſt the Term q in the ſecond Column will not be wanting. P q is alſo a Determinator of the transformed, for all the Terms of the transformed are below (or above) thoſe under P q, as were all the Terms of the propoſed Equation (Art. LXXXVI.); but it is a uſeleſs Determinator, becauſe P q being Part of P Q has the ſame Inclination to the Lines and Columns as P Q. Wherefore P Q having given $y = Ax^{b}$, P q will give $u = Bx^{b}$ (Art. LXXII.) conſequently having employed P Q, the Determinator P q is uſeleſs; but from the Point q there parts another Determinator which paſſes thro' the higheſt or loweſt full Cell of the firſt Column Q O, and which gives an Equation by which the ſecond Term of the Series is determined.




But if $y - Ax^{b} = 0$ be a multiple Root of the Equation furniſhed by the Determinator P Q, for Example, a triple Root; then in the transformed the Cells Q, q, q, remain empty, and the Determinator P q is terminated at the Cell R on the fourth Column, it is uſeleſs, for the Reaſons already given, to employ this Determinator. But there parts another (R S) from the Cell R, which may terminate in S at the firſt Column, and this is the ſecond Determinator which gives an Equation by which the ſecond Term of the Series will be found.

How the irregular terms of a ſeries are inveſtigated.

It may happen that the Term S is wanting, and that the Determinator R S is terminated on ſome other Column than the firſt, as in T, and then this Point T gives riſe to another Determinator T V; and if this does not reach the firſt Column, but is terminated ſooner, there parts from its Extremity another Determinator, and perhaps from that another, &c. to that one which is terminated on the firſt Column. Each

of thoſe Determinators being differently inclined to the Lines and Columns, give a different Exponent to the Power of $x$ in the ſecond (or third, fourth, &c.) Term; whereby the Series is forked into as many Series as there are Roots in all the Equations given by all thoſe Determinators.

To inveſtigate which, it ſuffices to conſider the Determinator $R\,T$, which parts from the Point $R$, Extremity of the firſt Determinator $P\,R$ which has been neglected, and to make Uſe of all the Roots of the Equation which it furniſhes; one of thoſe Roots is $u = 0$, the Sum of the Terms which are under this Determinator $R\,T$ being divifible by $u$, ſince $R\,T$ does not reach the firſt Column, which is the Line without $u$; and this Root being employed to obtain the following transformed Equation by ſubſtituting $0 + t$ for $u$, which is effected by only writing $t$ for $u$ in all the Terms of the Equation, which transformed conſequently furniſhes the ſame Determinators as the foregoing one. And as $P\,R$ and $R\,T$ have been already employed, we arrive at the Determinator $T\,V$ in the ſame Manner, as if at firſt we had paſſed from $P\,R$ to $T\,V$, the Root $u = 0$ of the Equation furniſhed by $R\,T$ producing the ſame Effect as if $R\,T$ had been neglected. It would be the ſame, if the Determinator $T\,V$ did not reach to the firſt Column, but was accompanied with other Determinators.

XCVII.

From whence it appears how the *irregular Terms* of the Series are to be calculated. I call by this Name thoſe which are given by Equations which have ſeveral Roots, real or imaginary. But it is ſcarce poſſible that this Kind of Diſorder ſhould laſt long; for as ſoon as we come to a Determinator whoſe Equation has no multiple Roots (or if a ſimple Root be employed, if this Equation has ſimple and multiple Roots) in the transformed Equation, there will be wanting of the Terms which are placed under this Determinator, for Example, $R\,S$, but the Term $S$ which ſhould be on the firſt Column (Art. XCV.). The Determinator therefore of this transformed will part from the Cell $T$ the higheſt (or loweſt) of the ſecond Column, and will paſs thro' the Cell $V$, alſo the higheſt (or the loweſt) of the full Cells in the firſt Column; wherefore, in the Equation furniſhed by this Determinator $T\,V$, the variable Quantity $u$ for Example does not ſurpaſs the firſt Degree, becauſe $T$ is on the ſecond Column, which is the Line $u$; and this Equation will only have one Root, which will certainly be real; and then the Series becomes regular, becauſe all the enſuing Determinators parting from the Point $T$, no Equations any more occurs which have ſeveral Roots; and all the ſucceeding Terms of the Series may be calculated with greater Facility by a Method we ſhall now proceed to explain.

How to know when a ſeries begins to become regular.

XCVIII.

Determination of the form of a regular series or the series of the exponents of the regular terms.

Suppose a Determinator to be obtained whose Equation has a simple Root, and that this Root is employed; for greater Simplicity I shall call it the first Determinator, abstracting from all the foregoing ones, if there were any; I shall also call the Equation which it furnishes the proposed one, altho' it may be one of the transformed ones. Let $m$ express the Order of the Terms thro' which this first Determinator passes, and let $y - Ax^{b} = 0$ be a simple Root of the Equation it furnishes. Substituting $Ax^{b} + u$ for $y$ in the Sum of the Terms of the Order $m$, the one which should be on the first Column would be $x^{m}$, we neglect its Coeficient as being of no Consequence; but this Term is wanting, since $y - Ax^{b}$ is supposed to divide the Sum of the Terms of the Order $m$ (Art. XCIII.). The Term $x^{m-b}u$ which follows falls on the second Column, and the Cell which it fills is the highest (or the lowest) of the full Cells of this Column. If $m \mp n$ is the Exponent of the Terms of the second Order (the Sign $-$ is for the descending Series, the Sign $+$ for the ascending ones) the highest (or the lowest) Term in the first Column will be $x^{m \mp n}$, hence the Determinator of this transformed passing thro' the Cells $x^{m-b}u$ and $x^{m \mp n}$ will give $u = Bx^{m \mp n - m + b} = Bx^{b \mp n}$ for the second Term of the Series. Wherefore the Difference $\mp n$ of the Exponents $h$ and $h \mp n$ of the first Term $Ax^{b}$, and of the second $Bx^{b \mp n}$, of the Series, is the same as that of the Exponents $m$ and $m \mp n$ of the first and second Order.

In all the following Transformations, the Cell $x^{m-b}u$ will remain full, $u$ being changed successively into $t$, $s$, $r$, &c. and all the following Determinators will part from this Cell to pass thro' the highest (or the lowest) of the full Cells of the first Column; they pass successively thro' the different Cells of this Column, because at each transformation, the Cell thro' which the Determinator passed is emptied; but on the other Hand, each transformation fills some new Cells of this first Column.

Hence when $Bx^{b \mp n} + t$ is substituted for $u$ in the first transformed, the Terms of the first Order $m$ fill the Cells $x^{m \mp n}$, $x^{m \mp 2n}$, $x^{m \mp 3n}$, $x^{m \mp 4n}$, &c. and the Terms of the second Order $m \mp n$ also fill the Cells

$x^{m \mp 2n}$, $x^{m \mp 3n}$, &c. in general $x^{m \mp jn}$ ($j$ expreſſing any Integer) but the ſame Subſtitution in the Terms of the Order $m + p$ will fill in the firſt Column, the Cells $x^{m \mp p}$, $x^{m \mp p \mp n}$, $x^{m \mp p \mp 2n}$, &c. in general $x^{m \mp p \mp jn}$.

If $x^{m \mp 2n}$ be the higheſt (or the loweſt) full Cell in the firſt Column, the third Determinator will paſs thro' $x^{m-b}$ $t$ and $x^{m \mp 2n}$, and will give $t = Cx^{b \mp 2n}$. And if afterwards $x^{m \mp 3n}$ is the higheſt (or the loweſt) of the full Cells of this Column, $s$ will be $= Dx^{b \mp 3n}$, and thus the ſucceſſive Exponents of $x$ in the Terms $y$, $u$, $t$, $s$, &c. of the Series, will be $h$, $h \mp n$, $h \mp 2n$, $h \mp 3n$, &c. in arithmetical Progreſſion, whoſe Difference is $n$. The Series would have no other Terms, if there were in the propoſed Equation no other Terms but thoſe of the Orders $m$ and $m + n$, all the Transformations, *ad infinitum*, would give no other Terms but ſuch as are contained under this general Expreſſion $x^{m \mp jn}$ ($j$ is an Integer, or ſometimes $= 0$). But if there be in the propoſed Equation, Terms of another Order, whoſe Exponent is $m + p$, the Cell $x^{m \mp p}$ will be once the higheſt (or the loweſt) of the firſt Column; then the Determinator, which parts always from the Cell $x^{m-b}$ $u$ (or $x^{m-b}$ $t$, or $x^{m-b}$ $s$, &c.) will give $u = Bx^{m \mp p - m \mp b}$ $= Bx^{b \mp p}$ (or $t = Cx^{b \mp p}$ or $s = Dx^{b \mp p}$; the Term in which $x$ has for Exponent $h \mp p$ is therefore one of the Terms of the Series.

The ſubſtitution of $Bx^{b \mp p} + t$ for $u$ (or $Cx^{b \mp p} + s$ for $t$, &c.) in the Terms of the Orders $m \mp jn$ will fill, in the firſt Column, the Cells $x^{m \mp jn \mp p}$, $x^{m \mp jn \mp 2p}$, $x^{m \mp jn \mp 3p}$, &c. this firſt Column therefore will acquire Terms included under the general Expreſſion $x^{m \mp jn \mp jp}$, and the Determinator paſſing ſucceſſively thro' thoſe Terms, will give to the Series, Terms contained under this general Expreſſion $Hx^{b \mp jn \mp jp}$.

If the propoſed Equation included a fourth Order of Terms whoſe Exponent is $m \mp q$, the general Expreſſion of the Terms of the Series would be $Hx^{b \mp jn \mp jp \mp jq}$, and ſo on, if there be a greater Number of Orders of Terms.

XCIX.

Hence, when in the Calculation of a Series we come to the *Regular Terms*, that is, when we come to a Determinator whoſe Equation has no multiple Roots, or when a ſimple Root of the Equation furniſhed by a Determinator is employed; the Succeſſion of the Exponents of $x$ in the following Terms of the Series are eaſily found thus: take the Exponents $m$, $m \mp n$, $m \mp p$, $m \mp q$, &c. of all the Orders of the Terms of the Equation, and ſubtract them all from the greater $m$ (or ſubtract from them all, the leaſt $m$) in order to obtain the Differences $n$, $p$, $q$, &c. then add or ſubtract from $h$, Exponent of the firſt Term, which is given by the firſt Determinator, the multiples $n$, $2n$, $3n$, &c. of the firſt Difference; then add to, or ſubtract from, all thoſe Terms ſucceſſively, the Multiples $p$, $2p$, $3p$, &c. of the ſecond Difference; and add to, or ſubtract from all thoſe Terms, the Multiples $q$, $2q$, $3q$, &c. of the third Difference ſucceſſively, and continue thus until all the Differences are exhauſted. Finally, range all thoſe Exponents according to their Magnitude.

Rule derived from the foregoing conſiderations for finding the ſucceſſion of exponents of the regular terms of a ſeries.

The arithmetical Progreſſion which begins by $h$, and whoſe Difference is the greateſt common Diviſor of $n$, $p$, $q$, &c. includes thoſe Exponents, but contains uſeleſs Terms, unleſs the leaſt Difference $n$ be a common Diviſor of all the other Differences.

C.

By this Rule, the Form of the Series is obtained, that is, the Succeſſion of the Powers of $x$ which form thoſe Terms; but the Coeficients of thoſe Terms are alſo to be found, which is eaſily effected, by giving to each of the Powers of $x$, which enters the Series, an undetermined Coeficient $A$, $B$, $C$, $D$, &c. and ſubſtituting, in the Equation, for $y$ this undetermined Series which expreſſes its Value, and determining, one after the other, each Coeficient $A$, $B$, $C$, $D$, &c. by means of the complex Coeficient of each Term of the transformed Equation, which are equal to Nothing, ſince otherwiſe the Whole could not be equal to Nothing.

CI.

Let us apply this Method to the Equation $6x^7 - 2x^5y^2 - a^3x^2y^2 + 4a^3x^3y + 2a^5xx - 3a^5xy + a^5yy = 0$, ſuppoſe the Value of $y$ in $x$, expreſſed by an aſcending Series, is required.

Application of this rule to an example.

The Equation being placed on the analytic Triangle, there is only one inferior Determinator, which gives the Equation $a^5yy - 3a^5xy + 2a^5xx = 0$, or $yy - 3xy + 2xx = 0$, which has two ſimple Roots $y = x$ and $y = 2x$, in general $y = Ax$, wherefore $h = 1$. Drawing ſtraight Lines parallel to the Determinator thro' all the Terms of the Equation, it will appear that it is compoſed of three Orders of Terms; the firſt, which contains the Terms $a^5yy$, $3a^5xy$, $2a^5xx$, thro' which paſſes the Determinator, has 2 for its Exponent, becauſe this ſtraight Line cuts the firſt Column in the Centre of the Cell $x^2$;

the ſecond includes the Terms $-a^3x^2y^2$ and $+4a^3x^3y$, and its Exponent is 4, becauſe the ſtraight Line which paſſes thro' the Cells $x^2y^2$ and $x^3y$ cuts the firſt Column in the Centre of the Cell $x^4$; and the third Order of Terms, which is compoſed of the Terms $-2x^5y^2$ and $6x^7$ has, for a ſimilar Reaſon, 7 for its Exponent: thoſe three Orders of Terms and their Exponents may be alſo diſcovered, by ſubſtituting in the Equation for $y$ its Value $\frac{b}{A}x = Ax$, which transforms it into $6x^7 - 2A^2x^7 - A^2a^3x^4 + 4Aa^3x^4 + 2a^5xx - 3Aa^5xx + A^2a^5xx = 0$, where it evidently appears, that the two firſt Terms are of the Order 7, the two following of the Order 4, and the three laſt of the Order 2. Since an aſcending Series is required, the leaſt Exponent 2 is to be ſubtracted from the two others 4 and 7, by which the Differences 2 and 5 are obtained; then the Numbers contained in the general Expreſſion $h + jn + jp$ or $1 + 2j + 5j$ is to be ſought, by adding to 1 ($h$), the Multiples of 2 ($n$), and adding to all thoſe Numbers the Multiples of 5 ($p$), which will give

1, 3, 5, 7, 9, 11, &c.
6, 8, 10, 12, 14, 16, &c.
11, &c.
16, &c.
21, &c.

which being ranged according to their Magnitude, conſtitute the Progreſſion 1, 3, 5, 6, 7, 8, 9, 10, 11, &c. the Form of the Series will be therefore $y = Ax + Bx^3 + Cx^5 + Dx^6 + Ex^7 + Fx^8 +$, &c. which Value of $y$ ſubſtituted in the Equation gives

$$
\begin{aligned}
6x^7 &= \ldots\ldots\ldots\ldots\ldots\ldots + 6x^7 \\
-2x^5y^2 &= \ldots\ldots\ldots\ldots\ldots\ldots - 2A^2x^7 \quad \bullet \quad - \&c. \\
-a^3x^2y^2 &= \ldots - a^3A^2x^4 - 2a^3ABx^6 - \bullet - 2a^3ACx^8 - \&c. \\
&\qquad\qquad\qquad\qquad\qquad\qquad - a^3B^2x^8 - \&c. \\
+4a^3x^3y &= \qquad + 4a^3Ax^4 + 4a^3Bx^6 + \bullet + 4a^3Cx^8 + \&c. \\
+2a^5xx &= +2a^5xx \\
-3a^5xy &= -3a^5Axx - 3a^5Bx^4 - 3a^5Cx^6 - 3a^5Dx^7 - 3a^5Ex^8 - \&c. \\
+a^5yy &= +a^5A^2xx + 2a^5ABx^4 + 2a^5ACx^6 + 2a^5ADx^7 + 2a^5AEx^8 + \&c. \\
&\qquad\qquad\qquad\qquad + a^5B^2x^6 + \bullet + 2a^5BCx^8 + \&c.
\end{aligned}
$$

Now since $6x^7 - 2x^5y^2 - a^3x^2y^2 + 4a^3x^3y + 2a^5xx - 3a^5xy + a^5yy = 0$, it follows, that the Sum of these Series involving $x$ must vanish; but that cannot be, if the Coeficient of every particular Term does not vanish. For every Term where $x$ is infinitely little, is infinitely greater than the following Terms: So that if every Term does not vanish of itself, the addition or subtraction of the following Terms, which are infinitely less than it, or of the preceding Terms, which are infinitely greater, cannot destroy it, and therefore the Whole cannot vanish. It appears therefore, that the complex Coeficient of each Term being put equal to nothing, will furnish the following Equations:

$2a^5 - 3a^5A + a^5AA = 0$, or $AA - 3A + 2 = 0$,
$-a^3A^2 + 4a^3A - 3a^5B + 2a^5AB = 0$,
or $AA - 4A = (2aaA - 3aa)B$,
$-2a^3AB + 4a^3B - 3a^5C + 2a^5AC + a^5B^2 = 0$,
or $2AB - 4B - aaBB = (2aaA - 3aa)C$,
$6 - 2A^2 - 3a^5D + 2a^5AD = 0$,
or $2AA - 6 = (2aaA - 3aa)a^3D$,
$-2a^3AC - A^3B^2 + 4a^3C - 3a^5E + 2a^5AE + 2a^5BC = 0$,
or $2AC + B^2 - 4C - 2a^2BC = (2a^2A - 3a^2)E$,
&c. &c.

for determining $A$, $B$, $C$, $D$, $E$, &c. and give

$A = 1$ . . . . . or $A = 2$
$B = \frac{AA - 4A}{2aaA - 3aa} = \frac{3}{aa}$ . . . . . $B = -\frac{4}{aa}$
$C = \frac{2AB - 4B - a^2B^2}{2aaA - 3aa} = \frac{15}{a^4}$ . . . . $C = -\frac{16}{a^4}$
$D = \frac{2AA - 6}{(2aaA - 3aa)a^3} = \frac{4}{a^5}$ . . . . $D = +\frac{2}{a^5}$
$E = \frac{2AC + BB - 4C - 2a^2BC}{2aaA - 3aa} = \frac{111}{a^6}$ $E = -\frac{112}{a^6}$
&c. &c.

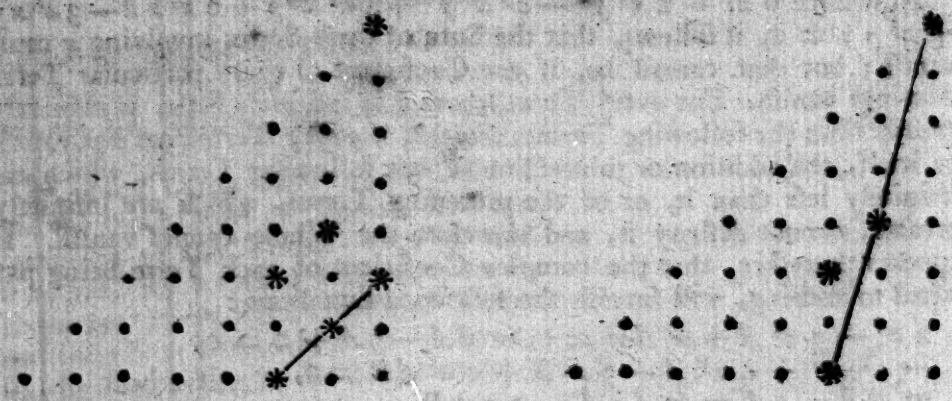
There are two ascending Series, $y = x + \frac{3x^3}{aa} + \frac{15x^5}{a^4} + \frac{4x^6}{a^5} + \frac{111x^7}{a^6}$, &c. and $y = 2x - \frac{4x^3}{aa} - \frac{16x^5}{a^4} + \frac{2x^6}{a^5} - \frac{112x^7}{a^6}$, &c.

CII.

Application of this rule to another example.

Let the Equation $x^7 - a^3x^3y + a^3x^2y^2 + a^5yy - 2a^5xy + a^5xx = 0$ be proposed, from which the Value of $y$ in $x$, expressed by an ascending Series, is required to be deduced.

This Equation being placed on the analytic Triangle, there will be found but one inferior Determinator, which gives the Equation $a^5yy - 2a^5xy + a^5xx = 0$ or $y^2 - 2xy + xx = 0$, which has



two equal Roots, or one double Root, $y = x$, whence $x + u$ being ſubſtituted for $y$ in the propoſed Equation, it will be transformed into $x^7 + a^3 x^3 u + a^3 x^2 u^2 + a^5 u u = 0$, which being in its turn placed on the analytic Triangle, there are two inferior Determinators, one of which gives the Equation $a^5 u u + a^3 x^3 u = 0$, or $u = -\frac{x^3}{aa}$, and the other gives the Equation $a^3 x^3 u + x_7 = 0$, or $u = -\frac{x^4}{a^3}$, both one and the other Exponent of $x$ exceeds the preceding one, conſequently thoſe two Determinators are uſeful; but it ſuffices to employ the firſt, which gives $u = -\frac{x^3}{a\,a} = A x^{b}$, that is, $h = 3$ and $A = -\frac{1}{a\,a}$, and ſubſtituting $A x^3$, or ſimply $x^3$, in the Equation $x^7 + a^3 x^3 u + a^3 x^2 u^2 + a^5 u u = 0$, it will be changed into $x^7 + a^3 x^6 + a^3 x^8 + a^5 x^6 = 0$, which, it is manifeſt, includes three Orders of Terms, whoſe Exponents are 6, 7, 8, and the Differences 1, 2. Since the leaſt divides the greateſt, the Succeſſion of the Exponents of $x$ will be the arithmetical Progreſſion 3, 4, 5, 6, &c. whoſe firſt Term is 3 ($h$), and the Difference 1 ($n$), the Form of the Series will be then $u = A x^3 + B x^4 + C x^5 + D x^6$, &c. and this Value of $x$ ſubſtituted in the Equation gives

$$
\begin{aligned}
x^7 &= +\ x^7 \\
+a^3x^3u &= +\ a^3 A x^6 + a^3 B x^7 + a^3 C x^8 + a^3 D x^9 + a^3 E x^{10} + \&c. \\
+a^3x^2u^2 &= +\ a^3 A^2 x^8 + 2\,a^3 A B x^9 + 2\,a^3 A C x^{10} + \&c. \\
&\quad +\ a^3 B B x^{10} + \&c. \\
+a^5 u u &= +a^5 A^2 x^6 + 2\,a^5 A B x^7 + 2\,a^5 A C x^8 + 2\,a^5 A D x^9 + 2\,a^5 A E x^{10} + \&c. \\
&\quad +\ a^5 B B x^8 + 2\,a^5 B C x^9 + 2\,a^5 B D x^{10} + \&c. \\
&\quad +\ a^5 C C x^{10} + \&c.
\end{aligned}
$$

From whence is deduced, by putting each complex Coeficient equal to Nothing,

$a^3 A + a^5 A A = 0$ . . . . . that is, $A = -\frac{1}{a a}$ or $A = 0$

$1 + a^3 B + 2 a^5 A B = 0$ . . . . . $B = \frac{1}{a^3}$ $B = -\frac{1}{a^3}$

$a^3 C + a^3 A A + 2 a^5 A C + a^5 B B = 0$ . . $C = \frac{2}{4a}$ $C = -\frac{1}{a^4}$

$a^3 D + 2 a^3 A B + 2 a^5 A D + 2 a^5 B C = 0$ $D = \frac{2}{a^5}$ $D = -\frac{2}{a^5}$

$a^3 E + 2 a^3 A C + a^3 B B + 2 a^5 A E + 2 a^5 B D + a^5 C C = 0$. $E = -\frac{7}{a^6}$ $E = -\frac{4}{a^6}$

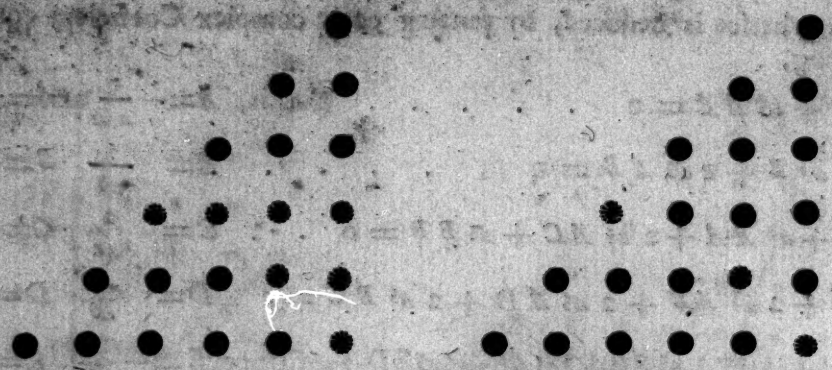
&c. &c. &c.

Hence $u$ is expreſſed by two Series, to each of which adding $x$, we will have two Values of $y = x - \frac{x^4}{a a} + \frac{x^4}{a^3} + \frac{2 x^5}{a^4} + \frac{2 x^6}{a^5} + \frac{7 x^7}{a^6}$, &c. and $y = x - \frac{x^4}{a^3} - \frac{x^5}{a^4} - \frac{2 x^6}{a^5} - \frac{4 x^7}{a^6}$, &c. this laſt Series is preciſely the ſame as that which would be obtained by the Means of the Determinator which gave the Equation $u = -\frac{x^4}{a^3}$ for ſubſtituting $x^4$ for $u$ in the Equation $x^7 + a^3 x^3 u + a^3 x^2 u^2 + a^5 u u = 0$, it will be changed into $x^7 + a^3 x^7 + a^3 x^{10} + a^5 x^8 = 0$, in which, it is manifeſt, there are three Orders of Terms, whoſe Exponents are 7, 8, 10, the Differences are therefore 1, 3; and as the leaſt divides the greateſt, the Succeſſion of the Exponents of $x$ is 4, 5, 6, 7, &c. and the Form of the Series $y = A x^4 + B x^5 + C x^6$, &c. whoſe Coeficients being determined, is converted into $-\frac{x^4}{a^3} - \frac{x^5}{a^4} - \frac{2 x^6}{a^5} - \frac{4 x^7}{a^6}$, &c. hence this Determinator gives only the ſecond of the two Series which the other had furniſhed.

CIII.

A deſcending Series is required, expreſſing the Value of $y$ in $x$, deduced from this Equation $x^2 y^3 + 3 a x^2 y^2 + 3 a^2 x^2 y + a^3 x x - a^3 x y - a^4 x + a^5 = 0$.

This Equation being placed on the analytic Triangle, furniſhes but one ſuperior Determinator, which gives the Equation $x^2 y^3 + 3 a x^2 y^3 + 3 a^2 x^2 y + a^3 x^2 = 0$, or dividing by $x^2$, $y^3 + 3 a y^2 + 3 a^2 y + a^3 = 0$, which has only one Root, but triple, $y = a$; ſubſtituting therefore $a + u$ for $y$ in the propoſed Equation, it will be transformed into $x^2 u^3 - a^3 x u + a^5 = 0$, which being placed on the analytic Triangle, furniſhes two ſuperior Determinators, one of which gives $x^2 u^3 - a^3 x u = 0$, or $u = \pm a^{3:2} x^{-1:2}$, and the other $-a^3 x u + a^5 = 0$, or $u = a^2 x^{-1}$.



But it suffices (Art. xcv.) to employ only the first, which gives $h = -\frac{1}{2}$, and substituting $x^{-1:2}$ for $u$ in the transformed Equation, it will change it into $x^{1:2} - a^3 x^{1:2} + a^5 x^0 = 0$, in which, it is manifest, there are only two Orders of Terms whose Exponents are $\frac{1}{2}$ and 0; there is therefore only one Difference $\frac{1}{2}$, the Succession of Exponents are therefore $-\frac{1}{2}, -1, -\frac{3}{2}, -2$, &c. and the Form of the Series $Ax^{-1:2} + Bx^{-1} + Cx^{-3:2}$, &c. this Series substituted for $u$ in

$$x^2u^3 = A^3x^{\frac{3}{2}} + 3AABx^0 + 3AACx^{-\frac{1}{2}} + 3AADx^{-1} + 3AAEx^{-\frac{3}{2}} + \&$$
$$+ 3ABB + 6ABC + 6ABD$$
$$+ 3ACC$$
$$+ B^3 + 3BBC$$

$$-a^3xu = -a^3Ax^{\frac{3}{2}} - a^3Bx^0 - a^3Cx^{-\frac{1}{2}} - a^3Dx^{-1} - a^3Ex^{-\frac{3}{2}} - \&c.$$
$$+a^5 = +a^5$$

furnishes the following Equations:

$A^3 - a^3A = 0$ . . . . . that is $A = \pm a^{3:2}$ or $A = 0$

$3AAB - a^3B + a^5 = 0$ . . . . . $B = -\frac{1}{2}aa$ $B = aa$

$3AAC + 3ABB - a^3C = 0$ . . . . . $C = \mp\frac{3}{8}a^{5:2}$ $C = 0$

$3AAD + 6ABC + B^3 - a^3D = 0$ . . $D = -\frac{1}{2}x^3$ $D = a^3$

$3AAE + 6ABD + 3ACC + 3BBC - a^3E = 0$ $E = \pm\frac{105}{128}a^{7:2}$ $E = 0$

&c. &c. &c.

Hence we have three descending Series expressing the Value of $u$ in $x$, viz. $u = a^{3:2}x^{-1:2} - \frac{1}{2}aax^{-1} - \frac{3}{8}a^{5:2}x^{-3:2} - \frac{1}{2}a^3x^{-2} + \frac{105}{128}a^{7:2}x^{-5:2}$, &c. $u = -a^{3:2}x^{-1:2} - \frac{1}{2}aax^{-1} + \frac{3}{8}a^{5:2}x^{-3:2} - \frac{1}{2}a^3x^{-2} - \frac{105}{128}a^{7:2}x^{-5:2}$, &c. $u = aax^{-1} + a^3x^{-2}$, &c. this last is precisely the same as would be obtained by the Means of the second Determinator, which gave $u = a^2x^{-1}$; for $x^{-1}$ substituted for $u$ in the Equation $x^2u^3 - a^3xu + a5$ transforms it into $x^{-1} - a^3 + a5 = 0$, which shews that there are two Orders of Terms, whose Exponents are $-1$ and $0$; there is therefore only one Difference 1, and the Form of the Series is $Ax^{-1} + Bx^{-2} + Cx^{-3} + Dx^{-4}$ + &c. which Value being substituted for $u$ in the Equation

$$x^2u^3 = A^3x^{-1} + 3AABx^{-2} + 3AACx^{-3} + 3AADx^{-4} + \text{&c.}$$
$$3ABB + 6ABC + B^3$$

$$-a^3xu = a^3A - a^3Bx^{-1} - a^3Cx^{-2} - a^3Dx^{-3} - a^3Ex^{-4} - \text{&c.}$$
$$+a5 = a5 + \text{&c.}$$

gives the following Equations:

| | |
|---|---|
| $-a^3A + a^5 = 0$ . . . . . | that is $A = a^2$ |
| $A^3 - a^3B = 0$ . . . . . | $B = a^3$ |
| $3AAB - a^3C = 0$ . . . . . | $C = 3a^4$ |
| $3AAC + 3ABB - a^3D = 0$ . . . . . | $D = 10a^5$ |
| $3AAD + 6ABC + B^3 - a^3E = 0$ . . | $E = 49a^6$ |
| &c. | &c. |

Wherefore $u = a^2x^{-1} + a^3x^{-2} + 3a^4x^{-3} + 10a^5x^{-4} + 49a^6x^{-5}$ &c. thus the Value sought of $y$ in $x$ is expressed by three descending Series, $y = a + a\sqrt{\frac{a}{x}} - \frac{aa}{2x} - \frac{3aa}{8x}\sqrt{\frac{a}{x}} - \frac{a^3}{2xx} + \frac{105a^3}{128xx}\sqrt{\frac{a}{x}}$ &c.

$y = a - a\sqrt{\frac{a}{x}} - \frac{aa}{2x} + \frac{3aa}{8x}\sqrt{\frac{a}{x}} - \frac{a^3}{2xx} - \frac{105a_3}{128xx}\sqrt{\frac{a}{x}}$ &c.

$y = a + \frac{aa}{x} + \frac{a^3}{xx} + \frac{3a^4}{x^3} + \frac{10a^5}{x4} + \frac{49a^6}{x5}$ &c.

CIV.

When in an indeterminate Equation, the Value of one of the variable Quantities ($y$) is found expreſſed by an infinite Series, including in its Terms the Powers of the other variable Quantity ($x$), and it is required to find the Value of $x$ in $y$, expreſſed by an infinite Series, whoſe Terms contain the Powers of $y$, this Method is called the Reverſion of Series, to which the preceding Doctrine may be conveniently applied.

Reverſion of ſeries.

Suppoſe we had found $y = ax + bx^2 + cx^3 + dx^4 + ex^5$, &c. whoſe Coeficients $a$, $b$, $c$, $d$, &c. are known, and the Value of $x$ in $y$ is required, I put $0 = y - +ax + bx^2 + cx^3 + dx^4$, &c. in which $x$ being ſuppoſed very ſmall, it will appear (Art. CXIX) that the Form of the Series muſt be $x = ly + my^2 + ny^3 + py^4$, &c. ſubſtituting this Value of $x$ in the Equation, we will have

$$
\begin{aligned}
-y &= -y \\
+ax &= +aly + amy^2 + any^3 + apy^4 + aqy^5 + ary^6 \text{ &c.} \\
+bx^2 &= \quad + bl^2y^2 + 2blmy^3 + bm^2y^4 + 2blpy^5 + bbny^6 \text{ &c.} \\
&\qquad + 2blny^4 + 2bmny^5 + 2blqy^6 \text{ &c.} \\
&\qquad\qquad + 2bmpy^6 \text{ &c.} \\
+cx^3 &= \quad + cl^3y^3 + 3cl^2my^4 + 3cl^2my^5 + cm^3y^6 \text{ &c.} \\
&\qquad + 3cl^2ny^5 + 6clmny^6 \text{ &c.} \\
&\qquad\qquad + 3cl^2py^6 \text{ &c.} \\
+dx^4 &= \quad + dl^4y^4 + 4dl^3my^5 + 6dl^2mmy^6 \text{ &c.} \\
&\qquad + 4dl^3ny^6 \text{ &c.} \\
+ex^5 &= \quad + el^5y^5 + 5el^4my^6 \text{ &c.} \\
+fx^6 &= \quad + fl^6y^6 \text{ &c.}
\end{aligned}
$$

which gives the the following Equations:

$al - 1 = 0$ . . . . . or $l = \frac{1}{a}$

$am + bll = 0$ . . . . . $m = -\frac{b}{a^3}$

$an + 2bml + cl^3 = 0$ . . . . $n = \frac{2bb - ac}{a^5}$

$ap + bm^2 + 2bln + 3cll m + dl^4 = 0, p = \frac{5abc - 5b^3 - a^2d}{a^7}$

&c. &c.

and ſubſtituting thoſe Values in $x = ly + my^2 + ny^3 + py^4$, &c. there will reſult $x = \frac{y}{a} - \frac{by^2}{a^3} + \frac{2bb - ac}{a^5}y^3 + \frac{5abc - 5b^3 - a^2d}{a^7}y^4 + \frac{14b^4 - 21acbb + 6a^2bd + 3a^2cc - ca^3}{a^9}y^5$, + &c. which is the Value required of $y$.

CCV.

When a Series is regular, that is, when a ſimple Root of the Equation furniſhed by the Determinator is employed, the Difference $(n)$ of the Exponents of $x$ in the firſt and the ſecond $(A x^{b}, B x^{b \mp n})$ Terms of the Series, is equal to the Difference of the Exponents $(m, m \mp n)$ of the firſt and ſecond Orders of Terms of the Equation; but it is not ſo if the Root $y - A x^{b} = 0$ be Multiple. If the Degree of its Multiplicity be expreſſed by $j$, that is, if $y - A x^{b}$ divides $j$ Times, the Sum of the Terms of the firſt Order $m$, provided it does not divide the Sum of the Terms of the ſecond Order $m \mp n$, the Difference $h - j$ of the Exponents of the firſt and ſecond Terms of the Series $A x^{b} + B x^{i}$, &c. will be equal to $\frac{n}{j}$, which is the Difference $n$ of the Exponents of the Orders $m$ and $\frac{m}{j} \mp n$ divided by $j$, ſo that in the ſecond Term, the Exponent of $x$ will be $(i =)\ h \mp \frac{n}{j}$.

Obſervation whereby may be diſcovered in many caſes if a ſeries is half imaginary.

For ſince $y - A x^{b} = 0$ is a Root whoſe Degree of Multiplicity is $j$, when $A x^{b} + u$ is ſubſtituted for $y$, there will be wanting in the transformed Equation the Terms correſponding to the Points where the Determinator cuts the $j$ firſt Columns, (Art. XCV.) that is, the Terms $x^{m}, x^{m-b} u, x^{m-2b} u^{2}$, &c. continued to the Term $x^{m-jb} u^{j}$, which will be the Term correſponding to the Extremity of this firſt Determinator; it is therefore from this Term that the ſecond Determinator parts, which will paſs thro' the firſt Term of the ſecond Order $x^{m \mp n}$ whoſe Cell will not be empty, ſince $y - A x^{b}$ is not a Diviſor of the Sum of the Terms of the ſecond Order. Hence the Equation which this ſecond Determinator furniſhes, will be of this Form $x^{m-jb} u^{j} = B^{j} x^{m \mp n}$ or $u^{j} = B^{j} x^{m \mp n - m + jb}$, or $u = B x^{b \pm \frac{n}{j}}$, the Exponent therefore of the ſecond Term is $h \mp \frac{n}{j}$.

Hence we may discover, without Calculation, whether a Series whose first Term is real, be half imaginary; when the Root $y - Ax^{\frac{b}{}} = 0$, furnished by the first Term, dividing several Times the Terms of the first Order $m$, does not divide those of the second Order $m \mp n$, which is a very ordinary Case. For if $j$, Degree of the multiplicity of this Root, is an even Number, and $n$, Difference of the Exponents of the Orders, an odd Number; the second Term $(Bx^{b \pm \frac{n}{j}})$ of the Series is half imaginary (Art. LXXXIII.) and it is real if $j$ be odd, but if $n$ and $j$ be both even, it will be real or imaginary, according as the Coeficients of the Terms $x^{m-jb}$ $u^{j}$ and $x^{m \mp n}$ have different Signs or the same Sign, but it is on this second Term it depends that the Series be real, or entirely or half imaginary, for the Equation which in the present Case furnishes this Term, consisting only of two Terms, will have no multiple Roots; from hence forth, the Series will become regular, and all its Terms, counting from the third, will be real.

END of SPECIOUS ARITHMETICK.

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## ERRATA.

Page 2. Art. II. Line 25. for, its triple more 410 makes 490, read, its triple more 410 makes 890.

P. 40. Art. LXIV, l. 9. for $ce = 18$, $bf = -12$, read, $ce = -12$, $bf = 18$.

P. 104. Art. LIX. l. 5. for $-3(BA' - AB')(EB' - BE')$ read, $-3(BA' - AB')(ED' - DE')$

P. 124. Art. X. l. 6. for, $\sqrt[3]{\frac{a^3c^5}{b}}x$, read, $\sqrt[3]{\frac{a^2c^5}{b}}x$. Line 7. for, $= a^{\frac{3}{2}}$, read, $a^{\frac{2}{3}}$. For, $b^{\frac{35}{12}}$, read, $c^{\frac{35}{12}}$.

Page 149. for, $\sqrt[3]{5} \times \sqrt[3]{20} \times \sqrt[3]{10}$, read, $\sqrt[3]{5} + \sqrt[3]{20} + \sqrt[3]{10}$.

P. 160. last Line, for, $ap - a = zp^{n}$, read, $ap - a = zp^{n} - z$. for $a +$, read $a \times$.

P. 223. l. 7. for, $\frac{AdA + BdB}{AA + BB}\times$, read, $\frac{AdA + BdB}{AA + BB}+$.

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